Mathematical Finance

Introduction to continuous time
Financial Market models

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Abstract

These are my Lecture Notes for a course in Continuous Time Finance which I taught in the Summer term 2003 at the University of Kaiserslautern. I am aware that the notes are not yet free of error and the manuscrip needs further improvement. I am happy about any comment on the notes. Please send your comments via e-mail to ce16@st-andrews.ac.uk.

Working Version
March 27, 2007

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Introduction

Mathematical Finance is the *mathematical* theory of financial markets. It tries to develop *theoretical* models, that *can* be used by "practitioners" to evaluate certain data from "real" financial markets. A model cannot be "right" or wrong, it can only be good or bad (for practical use). Even "bad" models can be "good" for theoretical insight.

Content of the lecture :

Introduction to continuous time financial market models.

We will give precise mathematical definitions, what we do understand under a *financial market*, until this let us think of a financial market as

some place where people can buy or sell financial derivatives.

During the lecture we will give various examples for financial derivatives. The following definition has been taken from [Hull] :

A *financial derivative* is a financial contract, whose value at expire is determined by the prices of the underlying financial assets (here we mean Stocks and Bonds).

We will treat options, futures, forwards, bonds etc. It is not necessary to have financial background.

During the course we will work with methods from

> *Probability theory*, *Stochastic Analysis* and *Partial Differential Equations*.

The Stochastic Analysis and Partial Differential Equations methods are part of the course, the Probability Theory methods should be known from courses like Probability Theory and Prama Stochastik.

Chapter 1

Stochastic Processes in Continuous Time

Given the present, the price S_t of a certain stock at some future time t is not known. We cannot look into the future. Hence we consider this price as a random variable. In fact we have a whole family of random variables S_t, for every future time t. Let's assume, that the random variables S_t are defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$, now it is time 0 , $0 \leq t < \infty$ and the σ-algebra $\mathcal{F}$ contains all possible information. Choosing sub σ-algebras $\mathcal{F}_t \subset \mathcal{F}$ containing all the information up to time t, it is natural to assume that S_t is $\mathcal{F}_t$ measurable, that is the stock price S_t at time t only depends on the past, not on the future. We say that $(S_t)_{t\in[0,\infty)}$ is $\mathcal{F}_t$ adapted and that S_t is a stochastic process. Throughout this chapter we assume that $(\Omega, \mathcal{F}, \mathbb{P})$ is a *complete probability space*. If X is a topological space, then we think of X as a measurable space with its associated *Borel σ-algebra* which we denote as $\mathcal{B}(X)$.

1.1 Filtrations and Stochastic Processes

Let us denote with I any subset of $\mathbb{R}$.

Definition 1.1.1. *A family $(\mathcal{F}_t)_{t\in I}$ of sub σ-algebras of $\mathcal{F}$ such that $\mathcal{F}_s \subset$*

$\mathcal{F}_t$ *whenever* $s < t$ *is called a* **filtration** *of* $\mathcal{F}$.

Definition 1.1.2. *A family* $(X_t, \mathcal{F}_t)_{t\in I}$ *consisting of* $\mathcal{F}_t$*-measurable* $\mathbb{R}^n$*-valued random variables* X_t *on* $(\Omega, \mathcal{F}, \mathbb{P})$ *and a Filtration* $(\mathcal{F}_t)_{t\in I}$ *is called an n-dimensional* **stochastic process**.

The case where $I = \mathbb{N}$ corresponds to stochastic processes in discrete time (see Probability Theory, chapter 19). Since this section is devoted to stochastic processes in continuous time, from now on we think of I as a connected subinterval of $\mathbb{R}_{\geq 0}$.

Often we just speak of the stochastic process X_t, if the reference to the filtration $(\mathcal{F}_t)_{t\in I}$ is clear. Also we say that $(X_t)_{t\in I}$ is $(\mathcal{F}_t)_{t\in I}$ *adapted*. If no filtration is given, we mean the stochastic process $(X_t, \mathcal{F}_t)_{t\in I}$ where

$$\mathcal{F}_t = \mathcal{F}_t^X = \sigma(X_s | s \in I, 0 \leq s \leq t)$$

is the σ-algebra generated by the random variables X_s up to time t.

Given a stochastic process $(X_t, \mathcal{F}_t)_{t\in I}$, we can consider it as function of two variables

$$\boxed{X : \Omega \times I \to \mathbb{R}^n\ ,\ (\omega, t) \quad \mapsto \quad X_t(\omega).}$$

On $\Omega \times I$ we have the product σ-algebra $\mathcal{F} \otimes \mathcal{B}(I)$ and for

$$I_t := \{s \in I | s \leq t\} \tag{1.1}$$

we have the product σ-algebras $\mathcal{F}_t \otimes \mathcal{B}(I_t)$.

Definition 1.1.3. *The stochastic process* $(X_t, \mathcal{F}_t)_{t\in I}$ *is called* **measurable** *if the associated map* $X : \Omega \times I \to \mathbb{R}^n$ *from (1.1) is* $(\mathcal{F} \otimes \mathcal{B}(I))/\mathcal{B}(\mathbb{R}^n)$ *measurable. It is called* **progressively measurable**, *if for all* $t \in I$ *the restriction of* X *to* $\Omega \times I_t$ *is* $(\mathcal{F}_t \otimes \mathcal{B}(I_t)/\mathcal{B}(\mathbb{R}^n)$ *measurable.*

In this course we will only consider measurable processes. So from now on, if we speak of a stochastic process, we mean a measurable stochastic process.

Working with stochastic processes the following space is of fundamental importance :

$$(\mathbb{R}^n)^I := Map(I, \mathbb{R}^n) = \{\omega : I \to \mathbb{R}^n\} \quad (1.2)$$

i.e. the maps from I to $\mathbb{R}^n$. For any $t \in I$ we have the so called *evaluation map*

$$\begin{aligned} ev_t : (\mathbb{R}^n)^I &\to \mathbb{R}^n \\ \omega &\mapsto \omega(t) \end{aligned}$$

Definition 1.1.4. *The σ-algebra on $(\mathbb{R}^n)^I$*

$$\sigma_{cyl} := \sigma(ev_s | s \in I)$$

generated by the evaluation maps is called the σ-algebra of **Borel cylinder sets**.

$$\mathcal{F}_t := \sigma_{cyl,t} := \sigma(ev_s | s \in I, s \leq t)$$

defines a filtration of σ_{cyl}. Whenever we consider $(\mathbb{R}^n)^I$ as a measurable space, we consider it together with this σ-algebra and this filtration.

The space $(\mathbb{R}^n)^I$ has some important subspaces :

$$\begin{aligned} C(I, \mathbb{R}^n) &:= \{\omega : I \to \mathbb{R}^n | \ \omega \text{ is continuous } \} &(1.3) \\ C_+(I, \mathbb{R}^n) &:= \{\omega : I \to \mathbb{R}^n | \ \omega \text{ is right-continuous } \} &(1.4) \\ C_-(I, \mathbb{R}^n) &:= \{\omega : I \to \mathbb{R}^n | \ \omega \text{ is left-continuous } \} &(1.5) \end{aligned}$$

Always, we consider these spaces as measurable spaces together with the associated σ-algebras of **Borel cylinder sets** and their filtration. These are defined as the corresponding restrictions of the σ-algebras from Definition 1.1.4. to these spaces.

In addition to (1.1) we can also consider the stochastic process $(X_t, \mathcal{F}_t)_{t\in I}$ as a map

$$X : \Omega \to (\mathbb{R}^n)^I \ , \ \omega \mapsto (t \mapsto X_t(\omega)). \qquad \textbf{(1.6)}$$

Exercise 1.1.1. *Show the map in (1.6) is $\mathcal{F}/\sigma_{cyl}$ measurable.*

Definition 1.1.5. *The stochastic process $(X_t, \mathcal{F}_t)_{t\in I}$ has* **continuous paths** *if $Im(X) \subset C(I, \mathbb{R}^n)$, where $Im(X)$ denotes the image of X. In this case, we often just say X is* **continuous**. *We say X has* **continuous paths almost surely** *if $\mathbb{P}\{\omega \in \Omega | X(\omega) \in C(I, \mathbb{R}^n)\} = 1$. X is called* **right-continuous** *if $Im(X) \subset C_+(I, \mathbb{R}^n)$ and in the same way as before has* **right-continuous paths almost surely** *if $\mathbb{P}\{\omega \in \Omega | X(\omega) \in C_+(I, \mathbb{R}^n)\} = 1$*

Via the map (1.6) under consideration of Exercise 1.1.1 the probability measure $\mathbb{P}$ on $(\Omega, \mathcal{F})$ induces a probability measure $\mathbb{P}^X$ on $(\mathbb{R}^n)^I$ which is also called *distribution* of X.

Definition 1.1.6. *Let $(X^i_t, \mathcal{F}^i_t)_{t\in I}$ $i = 1, 2$ be two stochastic processes defined on two not necessarily identical probability spaces $(\Omega^i, \mathcal{F}^i, \mathbb{P}_i)$. Then $(X^1_t)_{t\in I}$ and $(X^2_t)_{t\in I}$ are called* **equivalent** *if they have the same distribution, that is $\mathbb{P}_1^{X^1} = \mathbb{P}_2^{X^2}$. Often we write $(X^1_t)_{t\in I} \sim (X^2_t)_{t\in I}$.*

Clearly equivalence of stochastic processes is an equivalence relation. For a stochastic process $(X_t, \mathcal{F}_t)_{t\in I}$ consider the stochastic process $(ev^X_t)_{t\in I}$ on $((\mathbb{R}^n)^I, \sigma_{cyl}, \mathbb{P}^X)$ given by the evaluation maps. Then $(X_t)_{t\in I} \sim (ev^X_t)_{t\in I}$ and $(ev^X_t)_{t\in I}$ is called the *canonical representation* of $(X_t)_{t\in I}$. If $(X_t, \mathcal{F}_t)_{t\in I}$ has continuous paths then the stochastic process denoted by the same symbol $(ev^X_t)_{t\in I}$ on $(C(I, \mathbb{R}^N), \sigma_{cyl}, \mathbb{P}^X)$ is called the *canonical continuous representation* and clearly again $(X_t)_{t\in I} \sim (ev_t^X)_{t\in I}$.

Conclusion : If one is only interested in stochastic processes up to equivalence one can always think of the underlying probability space as $((\mathbb{R}^n)^I, \sigma_{cyl}, \mathbb{P}^X)$ or $(C(I, \mathbb{R}^N), \sigma_{cyl}, \mathbb{P}^X)$ in the continuous case. What *characterizes* the stochastic process is the probability measure $\mathbb{P}^X$.

In some cases though, equivalence in the sense of Definition 1.1.6 is not strong enough. The following definitions give stricter criteria on how to differentiate between stochastic processes.

Definition 1.1.7. *Let* $(X_t, \mathcal{F}_t)_{t\in I}$ $(Y_t, \mathcal{G}_t)_{t\in I}$ *be two stochastic processes on the* same *probability space* $(\Omega, \mathcal{F}, \mathbb{P})$. *Then* $(Y_t, \mathcal{G}_t)_{t\in I}$ *is called a* **modification** *of* $(X_t, \mathcal{F}_t)_{t\in I}$ *if*

$$P\{\ \omega | X_t(\omega) = Y_t(\omega)\ \} = 1\ ,\ \forall t \in I.$$

$(X_t, \mathcal{F}_t)_{t\in I}$ *and* $(Y_t, \mathcal{G}_t)_{t\in I}$ *are called* **indistinguishable***, if*

$$P\{\ \omega | X_t(\omega) = Y_t(\omega)\ \forall\ t \in I\ \} = 1.$$

The following two exercises are good to understand the relationships between equivalence, modification and indistinguishability.

Exercise 1.1.2. *Under the assumptions of Definition 1.1.7. Prove that the following implications hold :*

$(X_t, \mathcal{F}_t)_{t\in I}$ *and* $(Y_t, \mathcal{G}_t)_{t\in I}$ *indistinguishable* $\Rightarrow$
$(X_t, \mathcal{F}_t)_{t\in I}$ *and* $(Y_t, \mathcal{G}_t)_{t\in I}$ *are modifications of each other* $\Rightarrow$
$(X_t, \mathcal{F}_t)_{t\in I}$ *and* $(Y_t, \mathcal{G}_t)_{t\in I}$ *are equivalent.*

Give examples for the fact, that in general the inverse implication "⇐ " does not hold. But :

Exercise 1.1.3. *If in addition to the assumptions of Definition 1.1.7 we assume that $(X_t, \mathcal{F}_t)_{t\in I}$ and $(Y_t, \mathcal{G}_t)_{t\in I}$ are continuous and $(Y_t, \mathcal{G}_t)_{t\in I}$ is a modification of $(X_t, \mathcal{F}_t)_{t\in I}$, then $(X_t, \mathcal{F}_t)_{t\in I}$ and $(Y_t, \mathcal{G}_t)_{t\in I}$ are indistinguishable. How can the last two conditions be relaxed such that the implication still holds ?*

The last two definitions in this section concern the underlying filtrations.

Definition 1.1.8. *A filtration $(\mathcal{F}_t)_{t\in I}$ is called* **right-continuous** *if*

$$\mathcal{F}_t = \mathcal{F}_{t+} \quad := \bigcap_{s\in I, s>t} \mathcal{F}_s. \quad (1.7)$$

It is called **left-continuous** *if*

$$\mathcal{F}_t = \mathcal{F}_{t-} := \bigcup_{s\in I, s<t} \mathcal{F}_s. \quad (1.8)$$

Definition 1.1.9. *Let $I = [0, T]$ or $I = [0, \infty]$. A filtration $(\mathcal{F}_t)_{t\in I}$ satisfies the* **usual conditions** *if it is right continuous and $\mathcal{F}_0$ contains all $\mathbb{P}$ null-sets of $\mathcal{F}$.*

Exercise 1.1.4. *Let $I = [0, \infty)$. Show the filtration $(\sigma_{cyl,t})_{t\in I}$ of $(C(I, \mathbb{R}^n), \sigma_{cyl})$ is right-continuous as well as left-continuous.*

1.2 Special Classes of Stochastic Processes

There are two very important classes of stochastic processes, one is *martingales* the other is *Markov processes*, and there is the most important (continuous) stochastic process *Brownian motion* which belongs to both classes and will be treated in the next section. So far, let $(X_t, \mathcal{F}_t)_{t\in I}$ be a stochastic process defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

Definition 1.2.1. *If* $\mathbb{E}(|X_t|) < \infty \ \forall\, t \in I$ *then* $(X_t, \mathcal{F}_t)_{t\in I}$ *is called a*

1. **martingale** *if* $\forall s \leq t$ *we have* $\mathbb{E}(X_t|\mathcal{F}_s) = X_s$
2. **supermartingale** *if* $\forall s \leq t$ *we have* $\mathbb{E}(X_t|\mathcal{F}_s) \leq X_s$
3. **submartingale** *if* $\forall s \leq t$ *we have* $\mathbb{E}(X_t|\mathcal{F}_s) \geq X_s$

During the course we will see many examples of martingales as well as sub- and supermartingales.

Exercise 1.2.1. *Let* $(X_t, \mathcal{F}_t)_{t\in I}$ *be a stochastic process with independent increments, that means* $X_t - X_s$ *is independent of* $\mathcal{F}_u$ $\forall u \leq s$.*Consider the function* $\phi : I \to \mathbb{R}$, $\phi(t) = E(X_t)$. *Give conditions for* ϕ *that imply* X_t *is a martingale or submartingale or supermartingale.*

Exercise 1.2.2. *Let* Y *be a random variable defined on a complete probability space* $(\Omega, \mathcal{F}, \mathbb{P})$ *such that* $\mathbb{E}(|Y|) < \infty$ *and let* $(\mathcal{F}_t)_{t\in I}$ *be a filtration of* $\mathcal{F}$. *Define*

$$X_t := \mathbb{E}(Y|\mathcal{F}_t)\ ,\ \forall t \in I.$$

Show $(X_t, \mathcal{F}_t)_{t\in I}$ *is a martingale.*

For stochastic integration a class slightly bigger than martingales will play an important role. This class is called *local martingales*. To define it, we first need to define what we mean by a stopping time :

Definition 1.2.2. *A* **stopping time** *with respect to a filtration* $(\mathcal{F}_t)_{t\in I}$ *is an* $\mathcal{F}$ *measurable random variable* $\tau : \Omega \to I \cup \{\infty\}$ *such that for all* $t \in I$ *we have* $\tau^{-1}(I_t) \in \mathcal{F}_t$. *A stopping time is called* **finite** *if* $\tau(\Omega) \subset I$. *A stopping time is called bounded if there exists* $T^* \in I$ *such that* $\mathbb{P}\{\omega|\tau(\omega) \leq T^*\} = 1$.

The following exercises leads to many examples of stopping times.

Exercise 1.2.3. *Let* $(X_t, \mathcal{F}_t)_{t\in I}$ *be a continuous stochastic process with* *values in* $\mathbb{R}^n$ *and let* $A \subset \mathbb{R}^n$ *be a closed subset. Then*

$$\begin{aligned} \tau : \Omega &\rightarrow \mathbb{R} \\ \tau(\omega) &:= inf\{t \in I | X_t(\omega) \in A\} \end{aligned}$$

is a stopping time with respect to the filtration $(\mathcal{F}_t)_{t\in I}$.

Exercise 1.2.4. *Let* τ_1 *resp.*τ_2 *be stopping times on* $(\Omega, \mathcal{F}, \mathbb{P})$ *with respect to the filtrations* $(\mathcal{F}_t)_{t\in I}$ *resp.* $(\mathcal{G}_t)_{t\in I}$*. Let* $\mathcal{F}_t\mathcal{G}_t = \sigma(\mathcal{F}_t, \mathcal{G}_t)$*. Then*

$$\begin{aligned} \tau_1 \wedge \tau_2 : \Omega &\rightarrow \mathbb{R} \\ (\tau_1 \wedge \tau_2)(\omega) &= min(\tau_1(\omega), \tau_2(\omega)) \end{aligned}$$

is a stopping time with respect to the filtration $(\mathcal{F}_t\mathcal{G}_t)_{t\in I}$.

Given a stochastic process and a stopping time we can define a new stochastic process by stopping the old one. In case the stopping time is finite, we can define a new random variable. The definitions are as follows :

Definition 1.2.3. *Let* $(X_t, \mathcal{F}_t)_{t\in I}$ *be a stochastic process and* τ *a stopping time with respect to* $(\mathcal{F}_t)_{t\in I}$*. Then we define a new stochastic process* $(X_t^\tau)_{t\in I}$ *with respect to the same filtration* $(\mathcal{F}_t)_{t\in I}$ *as*

$$X_t^\tau(\omega) = \begin{cases} X_t(\omega) & ,\forall t \leq \tau(\omega) \\ X_{\tau(\omega)}(\omega) & ,\forall t > \tau(\omega) \end{cases} \quad (1.10)$$

If τ *is finite, we define a random variable* X_τ *on* $(\Omega, \mathcal{F}, \mathbb{P})$ *as*

$$X_\tau(\omega) := X_{\tau(\omega)}(\omega). \quad (1.12)$$

Also we can define a new σ-algebra :

Definition 1.2.4. *Let τ be a stopping time with respect to the filtration $(\mathcal{F}_t)_{t\in I}$. Then*

$$\mathcal{F}_\tau \ := \ \{A \in \mathcal{F} | A \cap \tau^{-1}(I_t) \in \mathcal{F}_t \, \forall \, t \in I\} \quad (1.14)$$

is called the σ-algebra of events up to time τ.

This is indeed a σ-algebra. The following is a generalization of Theorem 19.3 in the Probability Theory lecture.

Theorem 1.2.1. Optional Sampling Theorem *Let $(X_t, \mathcal{F}_t)_{t\in I}$ be a right-continuous martingale and τ_1,τ_2 be bounded stopping times with respect to $(\mathcal{F}_t)_{t\in I}$. Let us assume that $\tau_1 \leq \tau_2$ $\mathbb{P}$-almost sure. Then*

$$X_{\tau_1} = \mathbb{E}(X_{\tau_2} | \mathcal{F}_{\tau_1}). \qquad (1.16)$$

If $(X_t)_{t\in I}$ is only a submartingale (supermartingale) then (1.14) is still valid with $=$ replaced by $\leq$ ($\geq$).

Definition 1.2.5. *A stochastic process $(X_t, \mathcal{F}_t)_{t\in I}$ is called a* **local martingale** *if there exists an almost surely nondecreasing sequence of stopping times τ_n , $n \in \mathbb{N}$ with respect to $(\mathcal{F}_t)_{t\in I}$ converging to ∞ almost sure, such that $(X_t^{\tau_n}, \mathcal{F}_t)_{t\in I}$ is a martingale for all $n \in \mathbb{N}$.*

The class of local martingales contains the class of martingales. This follows from the Optional Sampling theorem. We leave the details as an exercise.

Exercise 1.2.5. *Show that every martingale is a local martingale.*

A relation between local martingales and supermartingales is established by the following :

Exercise 1.2.6. *A local martingale $(X_t, \mathcal{F}_t)_{t\in I}$ which is bounded below is a supermartingale. Bounded below means that there exists $c \in \mathbb{R}$ such that $\mathbb{P}\{\omega | X_t(\omega) \geq c \, , \, \forall t \in I\} = 1$.*

In plain words, the martingale property means, that the process, given the present time s has *no tendency* in future times $t \geq s$, that is the average over all future possible states of X_t gives just the present state X_s. In difference to this, the Markov property, which will follow in the next definition means that the process has *no memory*, that is the average of X_t knowing the past is the same as the average of X_t knowing the present. More precise :

Definition 1.2.6. $(X_t, \mathcal{F}_t)_{t \in I}$ *is called a* **Markov process** *if*

$$\mathbb{E}(X_t|\mathcal{F}_s) \quad = \quad \mathbb{E}(X_t|\sigma(X_s)) \; \forall \, 0 \leq s \leq t < \infty \qquad (1.18)$$

Sometimes the Markov property (1.16) is referred to as the *elementary Markov property*, in contrast to the *strong Markov property* which will be defined in a later section. So far, if we just say “Markov” we mean (1.16). Markov processes will arise naturally as the solutions of certain stochastic differential equations. Also Exercise 1.2.1 provides examples for Markov processes.

Besides martingales and Markov processes there is another class of processes which will occur from time to time in this text. It is the class of simple processes. This class is not so important for its own, but is important since its construction is simple and many other processes can be achieved as limits of processes from this class. Because of the simple construction, they are called simple processes.

Definition 1.2.7. *An n-dimensional stochastic process $(X_t, \mathcal{F}_t)_{t \in [0,T]}$ is called* **simple** *with respect to the filtration $(\mathcal{F}_t)_{t \in [0,T]}$ if there exist $0 = t_0 < t_1 < ... < T_m = T$ and $\alpha_i : \Omega \to \mathbb{R}^n$ such that α_0 is $\mathcal{F}_0$, α_i is $\mathcal{F}_{t_{i-1}}$ and*

$$X_t(\omega) = \alpha_0(\omega) \cdot 1_{\{0\}} + \sum_{i=1}^{m} \alpha_i(\omega) \cdot 1_{(t_{i-1}, t_i]}, \; \forall t, \omega. \qquad (1.19)$$

In case that in the definition above $m = 0$, we call $(X_t, \mathcal{F}_t)_{t \in [0,T]}$ a <u>constant</u> stochastic process.

1.3 Brownian Motion

Brownian Motion is widely considered as the most important (continuous) stochastic process. In this section we will give a short introduction into Brownian motion but we won't give a proof for its existence. There are many nice proofs available in the literature, but everyone of them gets technical at a certain point. So as in most courses about Mathematical Finance, we will keep the proof of existence for a special course in stochastic analysis.

Definition 1.3.1. *Let* $(W_t, \mathcal{F}_t)_{t\in[0,\infty)}$ *be an* $\mathbb{R}$*-valued continuous stochastic process on* $(\Omega, \mathcal{F}, \mathbb{P})$*. Then* $(W_t, \mathcal{F}_t)_{t\in[0,\infty)}$ *is called a* **standard Brownian motion** *if*

1. $W_0 = 0$ *a.s.*
2. $W_t - W_s \sim \mathcal{N}(0, t-s)$
3. $W_t - W_s$ *independent* $of \mathcal{F}_s$.

An $\mathbb{R}^n$ *valued process* $\mathbb{W}_t$ *is called an n-dimensional Brownian motion with initial value* $x \in \mathbb{R}^n$ *if*

$$\mathbb{W}_t \quad = \quad x + (W_t^1, ..., W_t^n),\ \forall t\ \in [0,\infty)$$

where W_t^i *are independent standard Brownian motions.*

It is not true that given a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$, there exists a standard Brownian motion or even an n-dimensional Brownian motion on this probability space, sometimes the underlying probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is just too small. Nevertheless the following is true :

Proposition 1.3.1. *There is a complete probability space* $(\Omega, \mathcal{F}, \mathbb{P})$ *such that there exists a standard Brownian motion* $(W_t, \mathcal{F}_t)_{t\in[0,\infty)}$ *on* $(\Omega, \mathcal{F}, \mathbb{P})$.

This process is unique up to equivalence of stochastic processes (see Definition 1.1.6)

Brownian motion can be used to build a large variety of martingales. One more or less simple way to to do this is given by the following exercise.

Exercise 1.3.1. *Let* $(W_t, \mathcal{F}_t)_{t\in[0,\infty)}$ *be a standard Brownian motion, and* $\sigma \in \mathbb{R}$ *a real number. For all* $t \in [0,\infty)$ *define*

$$X_t = e^{\sigma W_t - \frac{1}{2}\sigma^2 t}.$$

Then $(X_t, \mathcal{F}_t)_{t\in[0,\infty)}$ *is a (continuous) martingale.*

In the following we will take a closer look at the canonical continuous representation $(ev_t^W)_{t\in[0,\infty)}$ of any standard Brownian motion W (see page 6) defined on $(C([0,\infty),\mathbb{R}), \sigma_{cyl}, \mathbb{P}^W)$. The measure P^W is called the *Wiener measure*. Sometimes the Brownian motion W is also called *Wiener process*, hence the notation W.

For $t > 0$ consider the density functions (also called *Gaussian kernels*) of the standard normal distributions $\mathcal{N}(0,t)$ defined on $\mathbb{R}$ as

$$p(t,x) = \frac{1}{\sqrt{2\pi t}} e^{\frac{-x^2}{2t}}$$

The following proposition characterizes the Wiener measure.

Proposition 1.3.2. *The Wiener measure* $\mathbb{P}^W$ *on* $(C([0,\infty),\mathbb{R}), \sigma_{cyl})$ *is the unique measure which satisfies* $\forall m \in \mathbb{N}$ *and choices* $0 < t_1 < t_2 < ... < t_m < \infty$ *and arbitrary Borel sets* $A_i \in \mathcal{B}(\mathbb{R})$,$1 \leq i \leq m$

$$\mathbb{P}^W(\{\omega \in C([0,\infty),\mathbb{R}^n) |\, \omega(t_1) \in A_1, ..., \omega(t_m) \in A_m\}) =$$

$$\int_{A_1} p(t_1,x_1)dx_1 \int_{A_2} p(t_2 - t_1, x_2 - x_1)dx_2 ... \int_{A_m} p(t_m - t_{m-1}, x_m - x_{m-1})dx_m.$$

One proof of the existence of Brownian motion goes like this : Use the definition in Proposition 1.3.2 to define a measure on $((\mathbb{R}^n)^{[0,\infty)}, \sigma_{cyl})$. For this one needs the Daniell/Kolmogorov extension theorem ([Karatzas/Shreve], page 50). The result is a measure $\mathbb{P}^W$ and a process on $((\mathbb{R}^n)^{[0,\infty)}, \sigma_{cyl}, \mathbb{P}^W)$ which is given by evaluation and satisfies all the conditions in Definition 1.3.1 except the continuity. Then one uses the Kolomogorov/Centsov theorem ([Karatzas/Shreve], page 53) to show that this process has a continuous modification. This leads to a Brownian motion W_t.

Exercise 1.3.2. *Prove Proposition 1.3.2. Hint : Consider the joint density of* $(W_{t_0}, W_{t_1} - W_{t_0}, ..., W_{t_m} - W_{t_{m-1}})$ *of any standard Brownian motion* W *and use the density transformation formula (Theorem 11.4. in the Probability Theory Lecture) applied on the transformation* $g(x_1, ..., x_m) := (x_1, x_1 + x_2, ..., x_1 + x_2 + ... + x_m)$.

1.4 Black and Scholes' Financial Market Model

In this section we will introduce into the standard Black-Scholes model which describes the motion of a stock price and bond. First consider the following situation. At time $t = 0$ you put S_0^0 units of money onto your bank account and the bank has a constant deterministic interest rate $r > 0$. If after time $t > 0$ you want your money back, the bank pays you

$$S_t^0 = S_0^0 \cdot e^{rt}. \tag{1.20}$$

Let us consider the logarithm of this

$$ln(S_t^0) = ln(S_0^0) + rt. \tag{1.21}$$

So far there is no random, although in reality the interest rate is far from being constant in time and also nondeterministic. Now consider the price S_t^1 of a stock at time $t > 0$ and let S^1 denote the price at time $t = 0$. If we look at equation (1.19) the following approach seems to be natural

$$ln(S_t^1) = ln(S_0^1) + \tilde{b}t + \text{random}. \quad (1.22)$$

This equation means, that in addition to the linear deterministic trend in equation (1.19) we have some random fluctuation. This random fluctuation depends on the time t, hence we think of it as a stochastic process and denote it in the following with w_t. Since we know the stock price S_t^1 at time $t = 0$ there is no random at time $t = 0$ hence we can assume that

$$w_0 = 0 \text{ a.s .} \quad (1.23)$$

Furthermore we assume that w_t is composed of many similar small perturbations with no drift resulting from tiny little random events (events in the world we cannot foresee) all of which average to zero. The farther we look into the future, the more of these tiny little random events could happen, the bigger the variance of w_t is. We assume that the number of these tiny little events that can happen in the time interval $[0, t]$ is proportional to t. Hence by the central limit theorem an obvious choice for w_t is

$$w_t \sim \mathcal{N}(0, \sigma^2 t).$$

Since the number of tiny little events which can happen between time s and time t is proportional to $t - s$ we assume

$$w_t - w_s \sim \mathcal{N}(0, \sigma^2(t - s)) \text{ , } \forall\ s < t. \quad (1.24)$$

Also we assume once we know the stock price S_s^1 at some time $s > 0$ the future development S_t^1 for $t > s$ does not depend on the stock prices S_u^1 for $u < s$ before s. This translates to

$$w_t - w_s \text{ is independent of } w_u \text{ , } \forall u < s. \quad (1.25)$$

By comparison of equations (1.14)-(1.16) with (1 - 3) in definition

1.3.1 we must choose $w_t = \sigma W_t$, where $(W_t)_{t\in[0,\infty)}$ is a Brownian motion. We get the following equation for the stock price S_t^1

$$\boxed{S_t^1 = S_0^1 \cdot e^{\tilde{b}t+\sigma W_t}}$$

For applications it is useful to re-scale this equation using $b := \tilde{b} + \frac{1}{2}\sigma^2 t$ and writing

$$\boxed{S_t^1 = S_0^1 \cdot e^{(b-\frac{1}{2}\sigma^2)t+\sigma W_t}}$$

The model of a financial market consisting of one bond and one stock modeled as in equations (1.18) and (1.24) together with the appropriate set of trading strategies is called the *standard Black-Scholes model*. The valuation formula for Europeans call options in this model is called the *Black-Scholes* formula and was finally awarded with the Nobel-Prize in economics in 1997 for Merton and Scholes (Black was already dead at this time).

Exercise 1.4.1. *Let* $S_t = S_0 \cdot e^{(b-\frac{1}{2}\sigma^2)t+\sigma W_t}$ *denote the price of a stock in the standard Black-Scholes model. Compute the expectation* $\mathbb{E}(S_t)$ *and variance* $var(S_t)$.

Chapter 2

Financial Market Theory

In this section we will introduce into the theory of financial markets. The treatment here is as general as possible. At the end of this chapter we will consider the standard Black-Scholes model and derive the Black-Scholes formula for the valuation of an European call option. Throughout this chapter $(\Omega, \mathcal{F}, \mathbb{P})$ denotes a complete probability space and $I = [0, T]$ for $T > 0$.

2.1 Financial Markets

In the Introduction we gave a naive definition of what we think of a financial market is :

> **some place, where people can buy or sell financial derivatives**.

Hence we have to model two things. First the financial derivatives, second the actions (buy and sell) of the people (so called *traders) who take part in the financial market. The actions of the traders are henceforth called* trading strategies. Precisely :

Definition 2.1.1. *A* **financial market** *is a pair*

$$\mathcal{M}^{m,n} = ((X_t, \mathcal{F}_t)_{t\in I}, \Phi) \tag{2.1}$$

consisting of

1. *an $\mathbb{R}^{n+1}$ valued stochastic process $(X_t, \mathcal{F}_t)$ defined on $(\Omega, \mathcal{F}, \mathbb{P})$, such that $(\mathcal{F}_t)_{t\in I}$ satisfies the usual conditions and $\mathcal{F}_0 = \sigma(\emptyset, \Omega, \mathbb{P} -$ null-sets$)$*

2. *a set Φ which consists of $\mathbb{R}^{m+1}$ valued stochastic processes $(\varphi_t)_{t\in I}$ adapted to the same filtration $(\mathcal{F}_t)_{t\in I}$.*

The components $X_t^0, ..., X_t^m$ of X_t are called **tradeable** *components, the components $X_t^{m+1}, ..., X_t^n$ are called* **nontradeable**. *We denote the tradeable part of X_t with $X_t^{tr} = (X_t^0, ..., X_t^m)$. The elements of Φ are called* **trading strategies**.

The interpretation of Definition 2.1.1 is as follows : We think of the tradeable components as the evolution in time of assets which are traded at the financial market (for example stocks or other financial derivatives) and of the nontradeable components as additional (nontradeable !) parameter, describing the market. The set Φ is to be interpreted as the set of *allowed trading strategies. Sometimes $\phi \in \Phi$ is also called the* portfolio process. For a trading strategy $\varphi \in \Phi$ the i-th component φ_t^i denotes the amount of units of the i-th financial derivative owned by the trader at time t. In some cases we will assume that Φ carries some algebraic or topological structure, for example vector space (cone), topological vector space, L^2-process etc. We will specify this structure, when we really need it. The assumption on the 0-th filtration $\mathcal{F}_0$ makes sure, that any $\mathcal{F}_0$ measurable random variable on $(\Omega, \mathcal{F})$ is constant almost sure. This describes the situation that at time $t = 0$ we know completely what's going on.

Definition 2.1.2. *Let $\mathcal{M}^{m,n} = ((X_t, \mathcal{F}_t)_{t\in I}, \Phi)$ be a financial market and let $\varphi = (\varphi_t)_{t\in I} \in \Phi$ be a trading strategy, then we define the corresponding* **value process** *as*

$$V_t(\varphi) \quad = \varphi_t \cdot X_t^{tr} \quad = \sum_{i=0}^{m} \varphi_t^i X_t^i. \quad (2.2)$$

The value process gives us the worth of our portfolio at time t. In some cases it is helpful to consider the $(X_t^i)_{t\in I}$ in units of another stochastic process $(N_t, \mathcal{F}_t)_{t\in I}$ This leads to the notion of a numeraire.

Definition 2.1.3. *Consider a financial market* $\mathcal{M}^{m,n} = ((X_t, \mathcal{F}_t)_{t\in I}, \Phi)$. *A stochastic process* $(N_t, \mathcal{F}_t)_{t\in I}$ *is called a* **numeraire** *if it is strictly positive almost sure, that is*

$$\mathbb{P}\{\omega | N_t(\omega) > 0\} = 1 \ , \ \forall\, t \in I. \tag{2.3}$$

The numeraire is called a **market numeraire** *if there exists a trading strategy* $\varphi \in \Phi$ *such that* $(N_t)_{t\in I}$ *and* $(V_t(\varphi))_{t\in I}$ *are indistinguishable. Given a numeraire* $(N_t)_{t\in I}$ *we denote with*

$$\tilde{X}_t \quad := \quad \frac{X_t}{N_t}. \quad (2.4)$$

the **discounted price process** *and for any* $\varphi \in \Phi$

$$\tilde{V}_t(\varphi) \quad := \quad \frac{V_t(\varphi)}{N_t}. \quad (2.5)$$

the **discounted value process**.

As an example of a market numeraire one could think of a financial market where $(N_t)_{t\in[0,\infty)} = (X_t^0)_{t\in[0,\infty)}$ given by $X_t^0 = e^{rt}$ represents a bank account with deterministic interest rate $r > 0$. From now on, we will assume that there exists a market numeraire in the market $\mathcal{M}^{m,n} = ((X_t, \mathcal{F}_t)_{t\in I}, \Phi)$ and that it is given by the component of X_t^0 (the last thing is not really a restriction, think about it !) So in our case the component $\tilde{X}_t^0$ of the discounted price process is always constant equal to 1.

2.2 Arbitrage

In a financial market a risk free opportunity to make money is called an arbitrage. In our setting :

Definition 2.2.1. *An* **arbitrage** *in a financial market* $\mathcal{M}^{m,n} = ((X_t, \mathcal{F}_t)_{t\in I}, \Phi)$ *is a trading strategy* $\varphi \in \Phi$ *such that*

$$\boxed{\begin{array}{lcl} V_0(\varphi) & = & 0 \text{ almost sure} \\ V_T(\varphi) & \geq & 0 \text{ almost sure} \\ \mathbb{P}(V_T(\varphi) > 0) & > & 0 \end{array}}$$

$\mathcal{M}^{m,n} = ((X_t, \mathcal{F}_t)_{t\in I}, \Phi)$ *is called* **arbitrage free** *if there exist no arbitrages in* Φ.

Lemma 2.2.1. *Let* $\mathcal{M}^{m,n} = ((X_t, \mathcal{F}_t)_{t\in I}, \Phi)$ *be a financial market such that* Φ *contains all constant positive* $\mathbb{R}^{m+1}$*-valued processes and carries the algebraic structure of a cone. Then* $\mathcal{M}^{m,n}$ *is arbitrage free if there is no trading strategy* $\varphi \in \Phi$ *which satisfies*

$$\boxed{\begin{array}{lll} V_0(\varphi) & < 0 & \mathbb{P}\text{-almost sure} \\ V_T(\varphi) & \geq 0 & \mathbb{P}\text{-almost sure} \end{array}}$$

Proof. Assume that $\mathcal{M}^{m,n}$ is arbitrage free and there is a trading strategy $\varphi \in \Phi$ such that $V_0(\varphi) < 0$ and $V_T(\varphi) \geq 0$ $\mathbb{P}$-almost sure. Since $V_0(\varphi)$ is $\mathcal{F}_0$ we know that it is constant almost sure. Let c denote this constant. Then $c < 0$ and also $\tilde{c} = \frac{c}{X_0^0} < 0$. Using the assumption on Φ we can define a new trading strategy $\tilde{\varphi} \in \Phi$ as

$$\tilde{\varphi} = \begin{pmatrix} \varphi^0 - \tilde{c} \\ \varphi^1 \\ \cdot \\ \cdot \\ \varphi^m \end{pmatrix}.$$

It follows from the conditions on Φ in Definition 2.1.1 that $\tilde{\varphi}$ is again a trading strategy, i.e. $\tilde{\varphi} \in \Phi$. We have $V_0(\tilde{\varphi}) = V_0(\varphi) - \tilde{c}X_0^0 = V_0(\varphi) - V_0(\varphi) = 0$ almost sure and

$$V_T(\tilde{\varphi}) = V_T(\varphi) - \tilde{c}X_T^0.$$

Since $V_T(\varphi) \geq 0$, $\tilde{c} < 0$ and the numeraire $X_T^0 > 0$ almost sure, we have $V_T((\tilde{\varphi}) > 0$ almost sure. Hence $\mathbb{P}\{V_T((\tilde{\varphi}) > 0\} = 1 > 0$ and $\tilde{\varphi}$ is an arbitrage in $\mathcal{M}$. But this is a contradiction to the assumption that $\mathcal{M}$ is arbitrage free, hence such a φ can not exist in Φ.

□

Ideally a financial market is arbitrage free, but sometimes this is not the case. If arbitrages exist in a financial market, then mostly only for a short period of time. This is because if so, there are probably people who want to exploit the arbitrage and by exploitation of the arbitrage the arbitrage possibility vanishes. One can say, that the financial market has an arbitrage free equilibrium but sometimes differs from that equilibrium. The main implication of the no arbitrage condition in this general setup is given by the following proposition. It is often called the *No Arbitrage Principle*

Proposition 2.2.1. No Arbitrage Principle 1 *Let $\varphi, \psi \in \Phi$ be trading strategies in an arbitrage free financial market $\mathcal{M}^{m,n} = ((X_t, \mathcal{F}_t)_{t\in I}, \Phi)$ such that $V_T(\varphi) = V_T(\psi)$ $\mathbb{P}$-almost sure. If Φ is a vector space and contains all constant positive processes, then $V_0(\varphi) = V_0(\psi)$ $\mathbb{P}$-almost sure.*

Proof. Since $V_0(\varphi)$ and $V_0(\psi)$ are $\mathcal{F}_0$ measurable, they are constant almost sure. Let us assume $V_0(\varphi) \neq V_0(\psi)$, then w.l.o.g. $V_0(\varphi) < V_0(\psi)$ almost sure. Now consider the trading strategy $\varphi - \psi \in \Phi$. Then we have $V_t(\varphi - \psi) = V_t(\varphi) - V_t(\psi)$ for all $t \in [0, T]$. In particular $V_0(\varphi - \psi) < 0$ and $V_T(\varphi - \psi) = 0$ almost sure. From Lemma 2.2.1 it follows that $\mathcal{M}^{m,n}$ is not arbitrage free, which is a contradiction to the assumption that it is arbitrage free. □

Making more assumptions on the vector space Φ of trading strategies, one can indeed prove, that the two value processes $V_t(\varphi)$ and $V_t(\psi)$ are indistinguishable.

2.3 Martingale Measures

Whereas the measure $\mathbb{P}$ on the underlying measurable space $(\Omega, \mathcal{F})$ is somehow artificial and can be thought of as a subjective evaluation of the state of the financial market (for example from the point of view of one distinguished trader) martingale measures can be interpreted as an objective evaluation of the market. Therefore martingale measures are often used to price certain financial derivatives in a way, that no one can take advantage by trading these derivatives. In the right setup existence of martingale measures implies nonexistence of arbitrage. We will use martingale measures in the next section for the pricing of options and contingent claims.

Definition 2.3.1. *A probability measure $\mathbb{P}^*$ on $(\Omega, \mathcal{F}, \mathbb{P})$ is called an* **equivalent martingale measure** *for the financial market $\mathcal{M}^{m,n} = ((X_t, \mathcal{F}_t)_{t\in I}, \Phi)$, if*

1. *$\mathbb{P}$ and $\mathbb{P}^*$ have the same null-sets and*
2. *for any tradeable component X_t^i the discounted process $(\tilde{X}_t^i)_{t\in[0,T]}$ is a local $\mathbb{P}^*$ martingale.*

We denote the set of equivalent martingale measure for $\mathcal{M}^{m,n}$ as $\mathcal{P}(\mathcal{M}^{m,n})$.

Condition (2.) above is the same as saying $(\tilde{X}_t^{tr})$ is a local $\mathbb{P}^*$-martingale. The following definition seems to be very technical but in fact is quite useful when working with many measures at the same time.

Definition 2.3.2. *Let $g : \Omega \to \mathbb{R}$ be a real valued $\mathcal{F}$ measurable function. Then g is called* **universally integrable** *in the financial mar-*

ket $\mathcal{M}^{m,n} = ((X_t, \mathcal{F}_t)_{t\in I}, \Phi)$ *if for all equivalent martingale measures* $\mathbb{P}^* \in \mathcal{P}(\mathcal{M}^{m,n})$ *we have*

$$\mathbb{E}_{\mathbb{P}^*}(|g|) = \int_\Omega |g| d\mathbb{P}^* < \infty. \quad (2.6)$$

Often we just speak of a universally integrable function, though in fact we mean universally integrable in the financial market $\mathcal{M}^{m,n} = ((X_t, \mathcal{F}_t)_{t\in I}, \Phi)$, when the context is clear.

To establish the connection between martingale measures and arbitrage, we must restrict ourself to a special class of trading strategies which is from the real world financial market point of view very natural. Let us assume that at time t_0 a trader enters a market (only consisting of tradeable assets) and buys assets according to φ_{t_0}. Then the worth of his portfolio at time t_0 is

$$V_{t_0}(\varphi) = \varphi_{t_0} \cdot X_{t_0}.$$

Now he chooses not to change anything with his portfolio until time t_1. Then his portfolio at time t_1 still consists of φ_{t_0} and hence has worth

$$V_{t_1}(\varphi) = \varphi_{t_0} \cdot X_{t_1}. \quad (2.7)$$

At time t_1 though he chooses to rearrange his portfolio and reinvest all the money from $V_{t_1}(\varphi)$ according to φ_{t_1}. After this rearrangement the worth of his portfolio calculates as

$$V_{t_1}(\varphi) = \varphi_{t_1} \cdot X_{t_1}. \quad (2.8)$$

Since he only used the money from $V_{t_1}(\varphi)$ and didn't consume any of the money the two values from (2.7) and (2.8) must coincide. Hence we have

$$(\varphi_{t_1} - \varphi_{t_0}) \cdot X_{t_1} = 0, \tag{2.9}$$

or equivalently after division by the numeraire

$$(\varphi_{t_1} - \varphi_{t_0}) \cdot \tilde{X}_{t_1} = 0. \tag{2.10}$$

In the general framework of a financial market $\mathcal{M}^{m,n}$ trading at any time is allowed. For this purpose consider for any $t \in [0, T]$ partitions $\mathcal{Z}(t) : 0 = t_0 < t_1 < ... < t_k = t$ and define

$$\boxed{\begin{aligned} \sum_{\mathcal{Z}(t)} (d\varphi) \cdot \tilde{X} &= \sum_{i=0}^{k} (\varphi_{t_{i+1}} - \varphi_{t_i}) \cdot \tilde{X}^{tr}_{t_{i+1}} \\ \sum_{\mathcal{Z}(t)} \varphi \cdot d\tilde{X} &= \sum_{i=0}^{k} \varphi_{t_i} \cdot (\tilde{X}^{tr}_{t_{i+1}} - \tilde{X}^{tr}_{t_i}). \end{aligned}}$$

The concept above can then be generalized as follows : For any $t \in [0, T]$ and any sequence of partitions $\mathcal{Z}_l(t)$ such that $\lim_{l\to\infty} |\mathcal{Z}_l(t)| = 0$ we have

$$\lim_{l\to\infty} \sum_{\mathcal{Z}_l(t)} (d\varphi) \cdot \tilde{X} = 0 \text{ a.s } . \tag{2.11}$$

or equivalently since

$$\begin{aligned} \tilde{V}_t(\varphi) = \varphi_t \cdot \tilde{X}_t &= \tilde{V}_0(\varphi) + \sum_{i=0}^{k-1} (\varphi_{t_{i+1}} \cdot \tilde{X}^{tr}_{t_{i+1}} - \varphi_{t_i} \cdot \tilde{X}^{tr}_{t_i}) \\ &= \tilde{V}_0(\varphi) + \sum_{\mathcal{Z}(t)} (d\varphi) \cdot \tilde{X} + \sum_{\mathcal{Z}(t)} \varphi \cdot d\tilde{X}, \end{aligned}$$

$$\tilde{V}_0(\varphi) + \lim_{n\to\infty} \sum_{\mathcal{Z}_n(t)} \varphi \cdot d\tilde{X} = \tilde{V}_t(\varphi) \text{ a.s } . \tag{2.12}$$

This motivates the following definition of a self financing trading strategy.

Definition 2.3.3. *Let $\mathcal{M}^{m,n} = ((X_t, \mathcal{F}_t)_{t \in I}, \Phi)$ be a financial market and $\varphi \in \Phi$ a trading strategy .*

1. *$\varphi \in \Phi$ is called* **self financing** *if for any $t \in [0,T]$ and any sequence of partitions $\mathcal{Z}_l(t)$ such that $\lim_{l\to\infty} |\mathcal{Z}_l(t)| = 0$ we have*

$$\tilde{V}_0(\varphi) + \lim_{l\to\infty} \sum_{\mathcal{Z}_l(t)} \varphi \cdot d\tilde{X} \quad = \quad \tilde{V}_t(\varphi) \text{ a.s .} \qquad (2.13)$$

2. *φ is called* **strictly self financing** *if the convergence in (2.13) is locally dominated by universally integrable functions. This means that there are universally integrable functions g_j, a sequence of stopping times τ_j $j \in \mathbb{N}$ such that $\tau_j \leq \tau_{j+1}$, $\lim_{j\to\infty} \tau_j = \infty$ $\mathbb{P}$-almost sure and*

$$|\sum_{\mathcal{Z}_l(t)} \varphi^{\tau_j} \cdot d\tilde{X}^{\tau_j}| \leq g_j. \qquad (2.14)$$

We denote the set of strictly self financing trading strategies in Φ with Φ_s

The functions g_j in (2.14) may depend on the sequence of partitions $\mathcal{Z}_l(t)$. From the real world point of view the condition (2.14) is not so restrictive, since first of all, there is only finitely much money in the world at all, second there are financial (and other) restriction to the behavior of each individual trader.

Proposition 2.3.1. *Let $\varphi \in \Phi$ be a strictly self financing trading strategy in a financial market $\mathcal{M}^{m,n} = ((X_t, \mathcal{F}_t)_{t \in I}, \Phi)$ and $\mathbb{P}^* \in \mathcal{P}(\mathcal{M})$, then the discounted value process $\tilde{V}_t(\phi)$ follows a local $\mathbb{P}^*$-martingale.*

Proof. Since the nontradeable components of $(X_t)_{t\in[0,T]}$ have no effect on the value process we can assume that $m = n$, i.e. there are no nontradeable components. Also we can assume that $\tilde{V}_0(\varphi) = 0$. Let $\varphi \in \Phi$ be strictly self financing and τ_j' be a sequence of stopping times as in (2.14). Let τ_j'' be another sequence of stopping times as in Definition 1.2.5 such that corresponding to Definition 2.3.1 the stopped discounted price processes $\tilde{X}^{\tau_j''}$ are $\mathbb{P}^*$-martingales. Let us define new stopping times $\tau_j = \tau_j' \wedge \tau_j''$. Then $\tilde{X}^{\tau_j}$ are still $\mathbb{P}^*$- martingales (use the optional sampling theorem) and (2.14) is satisfied with τ_j instead of τ_j'. For $t \in [0,T]$ it follows from the theorem about dominated convergence and the strictly self financing condition that

$$\mathbb{E}_{\mathbb{P}^*}(|\tilde{V}_t^{\tau_j}(\varphi)|) \leq \mathbb{E}_{\mathbb{P}^*}(g_j) < \infty$$

where g_j is as in (2.14). Now let $0 \leq s \leq t \leq T$ and consider sequences of partitions

$$\begin{aligned}
\mathcal{Z}_l(s) &: \; 0 = t_0 < t_1 < ... < t_{k_l'} = s \\
\mathcal{Z}_l(s,t) &: \; s = t_{k_l'} < t_{k_l'+1} < ... < t_{k_l} = t \\
\mathcal{Z}_l(t) &: \; 0 = t_0 < t_1 < ... < t_{k_l'} = s < t_{k_l'+1} ... < t_{k_l} = t
\end{aligned}$$

such that $\lim_{l\to\infty} |\mathcal{Z}_l(t)| = 0$ and hence also $\lim_{l\to\infty} |\mathcal{Z}_l(s)| = 0$ and $\lim_{l\to\infty} |\mathcal{Z}_l(s,t)| = 0$. From the martingale property of $\tilde{X}^{\tau_j}$ it follows that

$$\mathbb{E}_{\mathbb{P}^*}(\tilde{X}_{t_{i+1}}^{\tau_j} - \tilde{X}_{t_i}^{\tau_j} | \mathcal{F}_{t_i}) = 0, \; \forall \, i.$$

Now let $i \geq k_l'$, then $t_i \geq s$ and we have

$$\mathbb{E}_{\mathbb{P}^*}(\varphi_{t_i}^{\tau_j} \cdot (\tilde{X}_{t_{i+1}}^{\tau_j} - \tilde{X}_{t_i}^{\tau_j} | \mathcal{F}_s) = \mathbb{E}_{\mathbb{P}^*}(\varphi_{t_i}^{\tau_j} \cdot \underbrace{\mathbb{E}_{\mathbb{P}^*}(\tilde{X}_{t_{i+1}}^{\tau_j} - \tilde{X}_{t_i}^{\tau_j} | \mathcal{F}_{t_i})}_{=0} | \mathcal{F}_s) = 0.$$

This implies

$$\mathbb{E}_{\mathbb{P}^*}(\sum_{\mathcal{Z}_l(s,t)} \varphi^{\tau_j} d\tilde{X}^{\tau_j} | \mathcal{F}_s) = 0, \; \forall\, l.$$

On the other side $\sum_{\mathcal{Z}_l(s)} \varphi^{\tau_j} d\tilde{X}^{\tau_j}$ is by definition $\mathcal{F}_s$ measurable, hence

$$\mathbb{E}_{\mathbb{P}^*}(\sum_{\mathcal{Z}_l(s)} \varphi^{\tau_j} d\tilde{X}^{\tau_j} | \mathcal{F}_s) = \sum_{\mathcal{Z}_l(s)} \varphi^{\tau_j} d\tilde{X}^{\tau_j}, \; \forall\, l.$$

Since $\sum_{\mathcal{Z}_l(t)} \varphi^{\tau_j} d\tilde{X}^{\tau_j} = \sum_{\mathcal{Z}_l(s)} \varphi^{\tau_j} d\tilde{X}^{\tau_j} + \sum_{\mathcal{Z}_l(s,t)} \varphi^{\tau_j} d\tilde{X}^{\tau_j}$ we conclude

$$\mathbb{E}_{\mathbb{P}^*}(\sum_{\mathcal{Z}_l(t)} \varphi^{\tau_j} d\tilde{X}^{\tau_j} | \mathcal{F}_s) = \sum_{\mathcal{Z}_l(s)} \varphi^{\tau_j} d\tilde{X}^{\tau_j}, \; \forall\, l.$$

Building the limit on both sides for $l \to \infty$ it follows again from the dominated convergence theorem and the strictly self financing condition that

$$\mathbb{E}_{\mathbb{P}^*}(\tilde{V}_t^{\tau_j}(\varphi) | \mathcal{F}_s) = \tilde{V}_s^{\tau_j}(\varphi)$$

which shows that $\tilde{V}_t^{\tau_j}(\varphi)$ follows a $\mathbb{P}^*$-martingale for all j. By the properties of the sequence of stopping times τ_j it follows that $\tilde{V}_t(\varphi)$ follows a local $\mathbb{P}^*$-martingale.

□

Though the self financing and strictly self financing condition seems to be very natural, it is however not so easy to determine whether a trading strategy $\varphi \in \Phi$ is self financing (strictly self financing) or not. The following Exercise shows that in the case of simple trading strategies this is much easier.

Exercise 2.3.1. *Let $\mathcal{M}^{m,n} = ((X_t, \mathcal{F}_t)_{t\in[0,T]}, \Phi)$ be a financial market such that Φ only consists of simple processes (see Definition 1.2.7). Show that if $\varphi \in \Phi$ is given as $\varphi_t(\omega) = \alpha_0(\omega) \cdot 1_{\{0\}} + \sum_{i=1}^m \alpha_i(\omega) \cdot 1_{(t_{i-1}, t_i]}, \; \forall t, \omega$ and a partition $0 = t_0 < t_1 < ... < t_m = T$, then φ is strictly self financing if and only if*

$$(\alpha_{t_{i+1}} - \alpha_{t_i}) \cdot X_{t_{i+1}}^{tr} = 0 \, \forall i,$$

To use the strictly self financing condition effectively, we have to introduce more notation.

Definition 2.3.4. *Let $\mathcal{M}^{m,n} = ((X_t, \mathcal{F}_t)_{t\in I}, \Phi)$ be a financial market. A trading strategy $\varphi \in \Phi$ is called* **tame** *if there exists $c \in \mathbb{R}$ such that $\tilde{V}_t(\varphi) \geq c$ $\mathbb{P}$-almost sure. We denote the set of tame trading strategies in Φ with Φ_t.*

The following Theorem shows us the first relation between arbitrage and martingale measures.

Theorem 2.3.1. *Let $\mathcal{M}^{m,n} = ((X_t, \mathcal{F}_t)_{t\in I}, \Phi)$ be a financial market and $\mathcal{P}(\mathcal{M}^{m,n}) \neq \emptyset$.Then the financial market $\mathcal{M}_{s,t}^{m,n} = ((X_t, \mathcal{F}_t)_{t\in I}, \Phi_s \cap \Phi_t)$ is arbitrage free. Here $\Phi_s \cap \Phi_t$ denotes the trading strategies in Φ which are strictly self financing and tame.*

Proof. Let $\varphi \in \Phi_s \cap \Phi_t$ such that $V_0(\varphi) = 0$ and $V_T(\varphi) \geq 0$ almost sure. Let $\mathbb{P}^* \in \mathcal{P}(\mathcal{M}^{m,n}) \neq \emptyset$. Since φ is strictly self financing it follows from Proposition 2.3.1 that $\tilde{V}_t(\varphi)$ follows a local $\mathbb{P}^*$ martingale. Since φ is tame and hence $\tilde{V}_t(\varphi)$ is bounded from below, it follows from Exercise 1.2.6 that $\tilde{V}_t(\varphi)$ follows a supermartingale. Therefore we have

$$\mathbb{E}_{\mathbb{P}^*}(\tilde{V}_T(\varphi)) = \mathbb{E}_{\mathbb{P}^*}(\tilde{V}_T(\varphi)|\mathcal{F}_0) \leq \tilde{V}_0(\varphi) = 0$$

Hence, since also $\tilde{V}_T(\varphi) \geq 0$ almost sure we have $\tilde{V}_T(\varphi) = 0$ almost sure. This then implies that $\mathbb{P}\{V_T(\varphi) > 0\} = 0$ and $\mathcal{M}_{s,t}^{m,n}$ is arbitrage free.

□

Example 2.3.1. Martingale Measure for the Black-Scholes Market *Let $\mathcal{M}^{1,1} = ((X_t, \mathcal{F}_t)_{t\in[0,T]}, \Phi)$ be the standard Black-Scholes model with an arbitrary set of trading strategies Φ modeled on the underlying probability space $(\Omega, \mathcal{F}, \mathbb{P})$ such that the numeraire $X_t^0 = e^{rt}$ represents a bank account and $X_t^1 = e^{(b-\frac{1}{2}\sigma^2)t+\sigma W_t}$ the price of a stock. We define a new probability measure $\mathbb{P}^*$ on $(\Omega, \mathcal{F})$ as follows : For any $A \in \mathcal{F}$ let*

$$\boxed{\mathbb{P}^*(A) \;:= \mathbb{E}_{\mathbb{P}}(Z_T \cdot 1_A) \;= \int_\Omega Z_T \cdot 1_A \quad \mathbf{(2.15)}}$$

where $Z_t := e^{\theta W_t - \frac{1}{2}\theta^2 t}$ *and* $\theta := \frac{r-b}{\sigma}$. θ *is called the* Market Price of Risk. *Then* $\mathbb{P}^*$ *is an equivalent martingale measure for the financial market* $\mathcal{M}^{1,1}$. *We postpone the proof of this until section 2.7. The ambitioned reader though should try to do the proof at this point. It is a very good exercise.*

In fact we would also like the discounted value process to be a (real) martingale, at least for some martingale measure $\mathbb{P}^* \in \mathcal{P}(\mathcal{M})$. This leads to the following definition :

Definition 2.3.5. *Let* $\mathcal{M}^{m,n} = ((X_t, \mathcal{F}_t)_{t \in I}, \Phi)$ *be a financial market. A strictly self financing and tame trading strategy* $\varphi \in \Phi$ *is called* **admissible** *if there exists* $\mathbb{P}^* \in \mathcal{P}(\mathcal{M})$ *such that the discounted value process* $\tilde{V}_t(\varphi)$ *is a martingale. We denote the set of admissible trading strategies with* Φ_a *and with* $\mathcal{M}_a^{m,n} = ((X_t, \mathcal{F}_t)_{t \in I}, \Phi_a)$ *the corresponding financial market.*

The following corollary follows directly from the definition of admissible and Theorem 2.3.1.

Corollary 2.3.1. *Let* $\mathcal{M}^{m,n} = ((X_t, \mathcal{F}_t)_{t \in I}, \Phi)$ *be a financial market and* $\mathcal{P}(\mathcal{M}^{m,n}) \neq \emptyset$. *Then the corresponding financial market* $\mathcal{M}_a^{m,n}$ *is arbitrage free.*

The following Theorem sharpens the No Arbitrage Principle in the presence of martingale measures.

Theorem 2.3.2. No Arbitrage Principle 2 *Let* $\mathcal{M}^{m,n} = ((X_t, \mathcal{F}_t)_{t \in I}, \Phi)$ *and* $\varphi, \psi \in \Phi_a$ *admissible trading strategies such that* $V_T(\varphi) = V_T(\psi)$. *Then the corresponding value processes* $V_t(\varphi) = V_t(\psi)$ *are indistinguishable.*

Proof. From the assumptions and the definition of admissibility it follow that there exist $\mathbb{P}_1^*, \mathbb{P}_2^* \in \mathcal{P}(\mathcal{M}^{m,n})$ such that $\tilde{V}_t(\varphi)$ follows a $\mathbb{P}_1^*$ martingale and $\tilde{V}_t(\psi)$ follows a $\mathbb{P}_2^*$ martingale. Because of the strictly self financing condition and tameness $\tilde{V}_t(\psi)$ also follows a $\mathbb{P}_1^*$ supermartingale. Hence

$$\tilde{V}_t(\varphi) = \mathbb{E}_{\mathbb{P}_1^*}(\tilde{V}_T(\varphi)|\mathcal{F}_t) = \mathbb{E}_{\mathbb{P}_1^*}(\tilde{V}_T(\psi)|\mathcal{F}_t) \leq \tilde{V}_t(\psi) \text{ a.s.}$$

Interchanging the roles of φ and ψ in the argumentation above completes the proof.

□

We have seen so far that the existence of an equivalent martingale measure for a financial market implies that it is arbitrage free. But does arbitrage freeness also imply the existence of an equivalent martingale measure ? In general not.

Definition 2.3.6. *A financial market* $\mathcal{M}^{m,n} = ((X_t, \mathcal{F}_t)_{t\in I}, \Phi)$ *satisfies the* **Fundamental Law of Asset Pricing** *if the following two conditions are equivalent :*

1. $\mathcal{M}^{m,n}$ *is arbitrage free.*
2. $\mathcal{P}(\mathcal{M}^{m.n}) \neq \emptyset$.

A *fundamental theorem of asset pricing* in this context is some kind of theorem which asserts that some class of financial markets satisfies the fundamental law of asset pricing. Most of the results so far have been established in the context of semi-martingales (see [Delbaen/Schachermayer] [Stricker]). In the context presented here, this is still up to further research.

Conjecture 1. Fundamental Theorem of Asset Pricing *Let* $\mathcal{M}^{m,n} = ((X_t, \mathcal{F}_t)_{t\in I}, \Phi)$ *be a financial market where* Φ *is an ample cone, then the corresponding market* $\mathcal{M}^{m,n} = ((X_t, \mathcal{F}_t)_{t\in I}, \Phi_a)$ *satisfies the fundamental law of asset pricing.*

2.4 Options and Contingent Claims

The reason for why mathematical advanced models of financial market have been developed is not only that some mathematician actually wanted to develop mathematical models, only they could understand, but more that people (traders) were in need for formulas to compute the right (fair) prices for what they called “options“.

Though options are traded on stock exchanges all over the world it is actually not so easy to describe mathematical what they are until one introduces a more general thing that is called a contingent claim. One could say an option is something which gives you the right (not the obligation) to buy some other thing at some time (in the future) for some predetermined price. To get a first idea consider the following example :

You want to give a grill party in about three weeks from now and therefore you need meat. Since you expect a lot of people you need a lot of meat. You decide to go to a butcher and ask how much you need and what is the price. The butcher tells you exactly how much you need, but he also tells you that obviously you cannot buy the meat today (because then it would be rotten in three weeks) and that he cannot say how much the meat will cost in three weeks. he tells you there might be another food scandal on its way and the meat prices could jump up and then you would have to pay much more than the price today. Nevertheless the butcher offers you the following : You pay him 5 Euro and then you can buy the meat in three weeks for todays price, even if the price in three weeks is much higher than today. Should you accept the offer, is 5 Euro a fair price ? Maybe the butcher tries to tricks you. What would be a fair price for such an offer ?

What is described above is what often is called an option. Let’s think about it like this. The Butcher offers you to buy the meat in three weeks from now for today's price, let's call this price K. Let S_T denote the price of the meat at time $T =$ 3 weeks . Then if $S_T \geq K$ you save $S_T - K$ Euro. If $S_T < K$ then you buy meat at the (spot) price in three weeks and save nothing. Since the price S_T is not known you consider this price as a random variable $S_T(\omega)$. The money you save can also be considered as a random variable via

$$g(\omega) = \begin{cases} S_T(\omega) - K, & \text{if } S_T \geq K \\ 0, & \text{if } S_T < K \end{cases} \tag{2.16}$$

So g can be thought of some random payment in the future. Something like this is mathematically known as a contingent claim.

Definition 2.4.1. *Let $\mathcal{M}^{m,n} = ((X_t, \mathcal{F}_t)_{t\in I}, \Phi)$ be a financial market. A* **contingent claim** *g is an $\mathcal{F}_T$-measurable random variable, such that*

$$g \geq 0 \text{ a.s. }, \exists\, \mu > 1 \text{ s.t. } \mathbb{E}(g^\mu) < \infty.$$

In the example above the intention to buy " the call on meat " was something like an insurance against the risk that the price of meat increases. This kind of behavior is called *Hedging*. Maybe some other people do not want to give a grill party, but are interested in some profit by trading in the meat market. They could act as follows. They buy the " call on meat " and hope that the meat price increases. Then at time T they buy the meat from the butcher at price K and sell it immediately back to the butcher at the price S_T and earn $S_T - K$. This kind of behavior is called *Speculation*.

Our aim in the next section will be to define the right prices for such contingent claim but before let us consider some types of contingent claims traded at financial markets with at least two tradeable assets (X_t^1) and (X_t^2). We have :

$$
\begin{aligned}
\text{European Call} &: (X_T^1 - K)^+ = max(S_T - K, 0) \\
\text{European Put} &: (K - X_T^1)^+ \\
\text{Call on maximum} &: (max(X_T^1, X_T^2) - K)^+ \\
\text{Call on average} &: (\int_0^T X_t^1 dt - K)^+ \\
\text{Down and out} &: (X_T^1 - K_1)^+ \cdot 1_{min_{0\le t\le T} X_t^1 > K_2}
\end{aligned}
$$

where K,K_1,K_2 are constants. K and K_1 are called strike prices, K_2 is a downside barrier and $K_1 > K_2$.These options are still of an elementary structure compared to other options traded at financial markets. People there are still inventing more and more complicated options, increasing the need for mathematician to evaluate these options. (Maybe the mathematician invent them themselves.)

2.5 Hedging and Completeness

The seller of a contingent claim must somehow make sure that at expiry time T he can fulfill his obligations. If he does this by investing in the financial market, then we speak of hedging.

Definition 2.5.1. *Let $\mathcal{M}^{m,n} = ((X_t, \mathcal{F}_t)_{t\in I}, \Phi)$ be a financial market and $g : \Omega \to \mathbb{R}$ be a contingent claim. A trading strategy $\varphi \in \Phi$ is called a*

1. **Hedging strategy** *for the contingent claim g if $V_T(\varphi) = g$ a.s.*
2. **Super Hedging strategy** *if $V_T(\varphi) \geq g$ a.s.*

Often we loosely speak of a **hedge** *respectively* **super hedge**, *meaning a hedging strategy respectively super hedging strategy. If there exists* *a hedging strategy for g then g is called Φ-***attainable***. If there exists a super hedging strategy for g then g is called Φ-***super attainable***.*

If there exist a hedging strategy for the contingent claim g then the seller of g can invest in the market corresponding to the trading strategy φ and makes sure, that at time T he can fulfill his obligations. Also, if there are two hedging strategies φ and ψ for the same contingent claim g, both belonging to Φ_a, the the value processes $(V_t(\varphi))_{t\in[0,T]}$ and $(V_t(\psi))_{t\in[0,T]}$ are indistinguishable by the No-Arbitrage Principle 2. However, the existence of such trading strategies in general is not guaranteed.

Definition 2.5.2. *A financial market $\mathcal{M}^{m,n} = ((X_t, \mathcal{F}_t)_{t\in I}, \Phi)$ is called* **complete***, if for any contingent claim $g : \Omega \to \mathbb{R}$ there exists a hedging strategy $\varphi \in \Phi$. Otherwise the market is called* **incomplete***.*

In the following we will see complete and incomplete markets, more incomplete markets in fact. We will see that some version of the standard Black-Scholes financial market model is complete. Incomplete markets are more difficult, we will see the reason for this in the next section. However incomplete markets arise quite naturally and the Black-Scholes model seems not to give the right picture for what is going on at real world financial markets. The algebraic structure of the set of contingent claims is that of a cone. It is however not clear, that if g_1 and g_2 are Φ-attainable, $g_1 + g_2$ is also Φ-attainable. To conclude this in general, we would need at least that Φ is a cone.

We saw in section 2.3. that arbitrage has something to do with the existence of martingale measure. Completeness has something to do with the uniqueness of martingale measure at least if one considers the following sub class of equivalent martingale measures :

Definition 2.5.3. *A probability measure $\mathbb{P}^* \in \mathcal{P}(\mathcal{M}^{m,n})$ is called a strong equivalent martingale measure for the financial market $\mathcal{M}^{m,n} = ((X_t, \mathcal{F}_t)_{t\in I}, \Phi)$ if for all $\varphi \in \Phi_a$ the discounted value process $\mathcal{V}_t(\varphi)$ follows a $\mathbb{P}^*$-martingale. We denote the set of strong equivalent martingale measures with $\mathcal{P}_s(\mathcal{M}^{m,n})$.*

The next theorem establishes the connection between completeness and uniqueness of equivalent martingale measures.

Theorem 2.5.1. (Uniqueness of equivalent Martingale Measures) *Let $\mathcal{M}^{m,n} = ((X_t, \mathcal{F}_t)_{t\in I}, \Phi_a)$ be a complete financial market, such that $\mathcal{F}_T = \mathcal{F}$, $\mathcal{P}(\mathcal{M}^{m,n}) \neq \emptyset$ and $\mathbb{E}_{\mathbb{P}}(X_T^0)^\mu < \infty$ for a $\mu > 0$. Then $|\mathcal{P}_s(\mathcal{M}^{m,n})| = 1$.*

Proof. Let $\mathbb{P}_1^*, \mathbb{P}_2^* \in \mathcal{P}_s(\mathcal{M}^{m,n})$ and $A \in \mathcal{F} = \mathcal{F}_T$. Then

$$g := 1_A \cdot X_T^0 : \Omega \to \mathbb{R}$$

is a contingent claim. Since $\mathcal{M}^{m,n}$ is complete there exists a hedge $\varphi \in \Phi_a$ such that $V_T(\varphi) = 1_A \cdot X_T^0$ almost sure. This of course implies $\tilde{V}_T(\varphi) = 1_A$. Hence we have

$$\mathbb{P}_i^*(A) = \mathbb{E}_{\mathbb{P}_i^*}(1_A|\mathcal{F}_0) = \mathbb{E}_{\mathbb{P}_i^*}(\tilde{V}_T(\varphi)|\mathcal{F}_0) = \tilde{V}_0(\varphi),\ i = 1, 2,$$

and in fact $\mathbb{P}_1^*(A) = \mathbb{P}_2^*(A)$.

□

2.6 Pricing of Contingent Claims

The pricing of contingent claims is one of the major topics of this course. By pricing we mean how to associate a price process to a contingent claim in a financial market, such that if this price process is considered as a tradeable asset (means the financial market is traded at the market), the corresponding extended market (with admissible trading strategies) is arbitrage free. For the whole section let $\mathcal{M}^{m,n} = ((X_t, \mathcal{F}_t)_{t\in[0,T]}, \Phi)$ a financial market.

Definition 2.6.1. *Let $g : \Omega \to \mathbb{R}$ be a contingent claim in $\mathcal{M}^{m,n}$. A price process $(g_t)_{t\in[0,T]}$ for g is a non negative $(\mathcal{F}_t)_{t\in[0,T]}$ adapted stochastic process such that $g_T = g$ almost sure.*

Before thinking about the question whether the extended financial market is arbitrage free or not, we must give a precise mathematical formulation of how to extend financial markets at all.

Definition 2.6.2. *Let* $\mathcal{M}^{m,n} = ((X_t, \mathcal{F}_t)_{t\in I}, \Phi)$ *and* $\mathcal{N}^{k,l} = ((Y_t, \mathcal{G}_t)_{t\in I}, \Psi)$ *be financial markets. Then we define the product of* $\mathcal{M}^{m,n}$ *and* $\mathcal{N}^{l,k}$ *as*

$$\boxed{\mathcal{M}^{m,n} \times \mathcal{N}^{l,k} \;\; := \;\; (\mathcal{M} \times \mathcal{N})^{m+k+1,n+l+1} = ((Z_t, \mathcal{H}_t)_{t\in I}, \Phi \times \Psi),}$$

with

$$Z_t \;\; = \;\; (X_t^0, ..., X_t^m, Y_t^0, ..., Y_t^k, X_t^{m+1}, ..., X_t^n, Y_t^{k+1}, ..., Y_t^l)^* \;\; \forall t,$$

$\mathcal{H}_t = \sigma(\mathcal{F}_t, \mathcal{G}_t)$ *and* $\Phi \times \Psi$ *the Cartesian product of the two sets of trading strategies.*

Given a contingent claim g and a price process $(g_t)_{t\in[0,T]}$ for this claim, we consider the financial market

$$\boxed{\mathcal{M}_{(g_t)} = ((g_t, \mathcal{F}_t)_{t\in[0,T]}, \Psi_{simple})}$$

where Ψ_{simple} denotes the one-dimensional simple processes (see Definition 1.2.7). We can now define the extended financial market, where discrete trading with the contingent claim is allowed as follows :

Definition 2.6.3. *Let* (g_t) *be a price process for a contingent claim* g *in a financial market* $\mathcal{M}^{m,n} = ((X_t, \mathcal{F}_t)_{t\in I}, \Phi)$*. We define the extended financial market as*

$$\boxed{\mathcal{M}^{m+1,n+1}(g_t) = \mathcal{M}^{m,n} \times \mathcal{M}_{(g_t)}}$$

where the product is given by (2.16).

Given a price process for a contingent claim, the contingent claim will only be traded if the price process gives no (significant) advantage to neither the seller or the buyer of the contingent claim. We would consider such a price as a “fair price“.

Definition 2.6.4. *A price process (g_t) for a contingent claim g in a financial market $\mathcal{M}^{m,n}$ is called a fair price if the extended financial market $\mathcal{M}_a^{m+1,n+1}(g_t)$ with admissible trading strategies is arbitrage free.*

Let us assume now that $\mathbb{P}^* \in \mathcal{P}(\mathcal{M}^{m,n})$ is an equivalent martingale measure and $g : \Omega \to \mathbb{R}$ is a contingent claim in this market. Then we can define

$$\boxed{g_t := X_t^0 \cdot \mathbb{E}_{\mathbb{P}^*}((X_T^0)^{-1} g | \mathcal{F}_t).}$$

Clearly g_t is a price process for the contingent claim g. The following theorem tells us that equivalent martingale measures compute fair prices for contingent claims.

Theorem 2.6.1. *Let $\mathcal{M}^{m,n} = ((X_t, \mathcal{F}_t)_{t\in I}, \Phi)$ be a financial market and g a contingent claim in this market. Let $\mathbb{P}^* \in \mathcal{P}(\mathcal{M}^{m,n})$ then the price process $g_t := X_t^0 \cdot \mathbb{E}_{\mathbb{P}^*}((X_T^0)^{-1} g|\mathcal{F}_t)$ is fair.*

Proof. We have to show that $\mathcal{M}_a^{m+1,n+1}(g_t)$ is arbitrage free .From Corollary 2.3.1. it follows that this is indeed the case if $\mathcal{P}(\mathcal{M}^{m+1,n+1}(g_t)) \neq \emptyset$. In fact we show that $\mathbb{P}^* \in \mathcal{P}(\mathcal{M}^{m+1,n+1}(g_t))$. But this is almost clear since by definition of an equivalent martingale measure, $\tilde{X}_t$ is a local $\mathbb{P}^*$-martingale and also $\tilde{g}_t = (X_t^0)^{-1} g_t = \mathbb{E}_{\mathbb{P}^*}((X_T^0)^{-1} g|\mathcal{F}_t)$ is a local $\mathbb{P}^*$-martingale (see also Exercise 1.2.2). □

In this way, martingale measures should be interpreted as pricing systems. They should be considered as (linear) functionals on the space of contingent claims with values in the space of fair prices. The interesting thing is, that is some cases if $|\mathcal{P}(\mathcal{M}^{m,n})| > 1$ not all of them give the same prices. It might be, that one equivalent martingale measure prices a contingent claim more expensive than another one, but still both prices are fair. Nevertheless the question arises, which of the equivalent martingale measures one should use to price contingent claims. Further research into this direction needs more advanced methods in functional analysis and topology then presented here.

A different method to price contingent claims is directly related to hedging strategies.

Definition 2.6.5. *let g be a contingent claim in a financial market $\mathcal{M}^{m,n} = ((X_t, \mathcal{F}_t)_{t\in I}, \Phi)$. We define the* **Hedging Price** *of g as*

$$\pi_{hedge} = inf\{x \in \mathbb{R} | \exists \text{ hedge } \varphi \in \Phi_a \text{ s.t. } V_0(\varphi) = x\}$$

and the **Super Hedging Price** *as*

$$\pi_{shedge} = inf\{x \in \mathbb{R} | \exists \text{ super hedge } \varphi \in \Phi_a \text{ s.t. } V_0(\varphi) = x\}$$

The hedging price tells us exactly the minimum investment at time 0 to get the pay out g at time T. This seems to be a quite reasonable procedure to price contingent claims. The question how these prices are related to martingale measures and whether they are fair or not will be answered by the following proposition and corollary.

Proposition 2.6.1. *If g is a contingent claim in $\mathcal{M}^{m,n} = ((X_t, \mathcal{F}_t)_{t\in I}, \Phi)$ and $\varphi \in \Phi_a$ a hedge for g, then the discounted value process $V_t(\varphi)$ is a fair price process for g. Furthermore we have*

$$V_t(\varphi) \geq X_t^0 \cdot \mathbb{E}_{\mathbb{P}^*}((X_T^0)^{-1} g), \ \forall\, \mathbb{P}^* \in \mathcal{P}(\mathcal{M}^{m,n}).$$

Proof. We have $g = V_T(\varphi)$ a.s. Since φ is admissible, there exists $\mathbb{P}^* \in \mathcal{P}(\mathcal{M}^{m,n})$ such that $\tilde{V}_T(\varphi)$ is a $\mathbb{P}^*$ martingale. Hence

$$\begin{aligned} V_t(\varphi) = \quad X_t^0 \cdot \tilde{V}_t(\varphi) \quad &= X_t^0 \cdot \mathbb{E}_{\mathbb{P}^*}(\tilde{V}_T(\varphi)|\mathcal{F}_t) \\ &= X_t^0 \cdot \mathbb{E}_{\mathbb{P}^*}((X_T^0)^{-1} V_T(\varphi)|\mathcal{F}_T) \\ &= X_t^0 \mathbb{E}_{\mathbb{P}^*}((X_T^0)^{-1} g|\mathcal{F}_t) \end{aligned}$$

and $V_t(\varphi)$ is a fair price process by Theorem 2.6.1. For any other equivalent martingale measure $\tilde{\mathbb{P}^*} \in \mathcal{P}(\mathcal{M}^{m,n})$ $\tilde{V}_t(\varphi)$ follows a $\tilde{\mathbb{P}^*}$-super martingale (local martingale and bounded from below !) and hence

$$\begin{aligned} V_t(\varphi) = \quad X_t^0 \cdot \tilde{V}_t(\varphi) \quad & \geq X_t^0 \cdot \mathbb{E}_{\tilde{\mathbb{P}^*}}(\tilde{V}_T(\varphi)|\mathcal{F}_t) \\ & = X_t^0 \cdot \mathbb{E}_{\tilde{\mathbb{P}^*}}((X_T^0)^{-1} V_T(\varphi)|\mathcal{F}_T) \\ & = X_t^0 \mathbb{E}_{\tilde{\mathbb{P}^*}}((X_T^0)^{-1} g|\mathcal{F}_t) \end{aligned}$$

□

Corollary 2.6.1. *Let g be a contingent claim in $\mathcal{M}^{m,n}$ then*

$$\pi_{hedge} \geq X_0^0 \cdot \mathbb{E}_{\mathbb{P}^*}((X_T^0)^{-1} g), \; \forall \, \mathbb{P}^* \in \mathcal{P}(\mathcal{M}^{m,n}).$$

2.7 The Black-Scholes Formula

In this section we restrict ourself to one distinguished financial market model, the standard Black-Scholes model of section 1.4. and Example 2.3.1, and one distinguished contingent claim, the European call option from section 2.5. The model is given within a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ which carries a standard Brownian motion $(W_t, \mathcal{F}_t)$ by a two dimensional price process

$$(X_t)_{t\in[0,T]} = \begin{pmatrix} X_t^0 \\ X_t^1 \end{pmatrix}_{t\in[0,T]} = \begin{pmatrix} e^{rt} \\ S_0 \cdot e^{(b-\frac{1}{2}\sigma^2)t+\sigma W_t}1 \end{pmatrix}_{t\in[0,T]}$$

where X_t^0 represents a bank account with deterministic interest rate $r > 0$ and X_t^1 the price of a stock with initial price $S_0 > 0$. We do not fix the set Φ of trading strategies here, but we think of the set of simple processes as allowed trading strategies. This is reasonable at least, since at real world markets, trading only happens in discrete time steps. We denote this market with $\mathcal{M}_{BS}^{1,1}$. In Example 2.3.1 we defined a new measure $\mathbb{P}^*$on $(\Omega, \mathcal{F})$ as

$$\boxed{\mathbb{P}^*(A) \;:=\; \mathbb{E}_{\mathbb{P}}(Z_T \cdot 1_A) = \int_\Omega Z_T \cdot 1_A}$$

where $Z_t := e^{\theta W_t - \frac{1}{2}\theta^2 t}$ and $\theta := \frac{r-b}{\sigma}$. Since $Z_T > 0$ it is clear that $\mathbb{P}$ and $\mathbb{P}^*$ are equivalent measures. Let us now show that the process defined by

$$W_t^* := W_t - \theta \cdot t, \; \forall\, t \tag{2.17}$$

is a Brownian motion with respect to the measure $\mathbb{P}^*$. Clearly $W_0^* = W_0 = 0$ $\mathbb{P}$- and hence also $\mathbb{P}^*$-almost sure. Also $W_t^* - W_s^* = W_t - W_s - \theta(t-s)$ is independent of $\mathcal{F}_s$. Let us now compute the distribution function of $W_t^* - W_s^*$. Notice first, that if $A \in \mathcal{F}_t$, we have

$$\mathbb{P}^*(A) = \mathbb{E}_{\mathbb{P}}(1_A \cdot Z_T) = \mathbb{E}_{\mathbb{P}}(1_A \cdot \mathbb{E}_{\mathbb{P}}(Z_T|\mathcal{F}_t) = \mathbb{E}_{\mathbb{P}}(1_A \cdot Z_t),$$

since $(Z_t)_{t\in[0,T]}$ is a $\mathbb{P}$-martingale (see Exercise 1.3.1). We have

$$\begin{aligned}
\mathbb{P}^*(W_t^* - W_s^* \le x) &= \mathbb{E}_{\mathbb{P}}(1_{\{W_t^* - W_s^* \le x\}} \cdot Z_t) \\
&= \mathbb{E}_{\mathbb{P}}(\mathbb{E}_{\mathbb{P}}(1_{\{W_t^* - W_s^* \le x\}} \cdot Z_t Z_s^{-1} | \mathcal{F}_s) Z_s) \\
&= \mathbb{E}_{\mathbb{P}}(\mathbb{E}_{\mathbb{P}}(\underbrace{1_{\{W_t^* - W_s^* \le x\}} \cdot e^{\theta(W_t - W_s) - \frac{1}{2}\theta^2(t-s)}}_{\text{independent of } \mathcal{F}_s} | \mathcal{F}_s) Z_s) \\
&= \mathbb{E}_{\mathbb{P}}(1_{\{W_t - W_s \le x + \theta(t-s)\}} \cdot e^{\theta(W_t - W_s) - \frac{1}{2}\theta^2(t-s)}) \cdot \underbrace{\mathbb{E}_{\mathbb{P}}(Z_s)}_{=1}
\end{aligned}$$

Using the density function of $W_t - W_s \sim \mathcal{N}(0, t-s)$ (with respect to the measure $\mathbb{P}$)we get that the last expression above is equal to

$$\begin{aligned}
& \int_{-\infty}^{x+\theta(t-s)} \frac{1}{\sqrt{2\pi(t-s)}} e^{\theta y - \frac{1}{2}\theta^2(t-s)} \cdot e^{-\frac{y^2}{2(t-s)}} dy \\
\underbrace{=}_{z = y - \theta(t-s)} & \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi(t-s)}} e^{\theta z + \frac{1}{2}\theta^2(t-s)} \cdot e^{-\frac{(z+\theta(t-s))^2}{2(t-s)}} dz
\end{aligned}$$

In the last integral almost everything cancels, and what is left is the expression

$$\int_{-\infty}^{x} \frac{1}{\sqrt{2\pi(t-s)}} e^{-\frac{z^2}{2(t-s)}}\,dy.$$

But this (as a function of x) is the distribution function of an $\mathcal{N}(0, t-s)$ distributed random variable. Hence we have proven that $W_t^* - W_s^* \sim \mathcal{N}(0, t-s)$ under the measure $\mathbb{P}^*$. Since obviously the process $(W_t^*)_{t\in[0,T]}$ has continuous paths, we have proven that it is a standard Brownian motion under $\mathbb{P}^*$.

Clearly $\tilde{X}_t^0 \equiv 1$ is a martingale under $\mathbb{P}^*$. Let us now consider $X_t^1 = x \cdot e^{(b-\frac{1}{2}\sigma^2)t+\sigma W_t}$. By a simple transformation we get

$$X_t^1 = S_0 \cdot e^{(b-\frac{1}{2}\sigma^2)t+\sigma W_t} = x \cdot e^{\sigma W_t-(r-b)t+rt-\frac{1}{2}\sigma^2 t} = S_0 \cdot e^{\sigma W_t^* -(r-\frac{1}{2}\sigma^2)t}.$$

$$\tilde{X}_t^1 = e^{-rt} X_t^1 = x \cdot e^{\sigma W_t^* - \frac{1}{2}\sigma^2 t}.$$

Since $(W_t^*)_{t\in[0,T]}$ is a $\mathbb{P}^*$ Brownian motion the last expression follows a $\mathbb{P}^*$-martingale (see again Exercise 1.3.1). Hence we have proven that $\mathbb{P}^*$ is an equivalent martingale measure for the market $\mathcal{M}_{BS}^{1,1}$ and that $(\mathcal{M}_{BS}^{1,1})_a$ is arbitrage free. Let us now consider a European call option with strike price K given by

$$\begin{array}{|rcl|} g : \Omega & \to & \mathbb{R} \\ g(\omega) & = & (X_T^1(\omega - K)^+. \end{array}$$

Using Theorem 2.6.1 we can compute a fair price for this option as

$$\begin{aligned} g_0 &= \mathbb{E}_{\mathbb{P}^*}(e^{-rT} \cdot (X_T^1 - K) \cdot 1_{\{X_T^1 \geq K\}}) \\ &= \underbrace{\mathbb{E}_{\mathbb{P}^*}(e^{-rT} X_T^1 \cdot 1_{\{X_T^1 \geq K\}})}_{=(I)} - \underbrace{e^{-rT} K \cdot \mathbb{P}^*(X_T^1 \geq K)}_{=(II)} \end{aligned}$$

Let us compute the expressions (I) and (II) :

$$
\begin{aligned}
(I) &= \mathbb{E}_{\mathbb{P}^*}(S_0 e^{\sigma W_T^* - \frac{1}{2}\sigma^2 T} \cdot 1_{\{ln(S_0)+(r-\frac{1}{2}\sigma^2)T+\sigma W_T^* \geq ln(K)\}} \\
&= S_0 \cdot \int_{\frac{ln(\frac{K}{S_0})-(r-\frac{1}{2}\sigma^2)T}{\sigma}}^{\infty} \frac{1}{\sqrt{2\pi T}} e^{\sigma x - \frac{1}{2}\sigma^2 T} \cdot e^{-\frac{x^2}{2T}} dx \\
&= S_0 \cdot \int_{\frac{ln(\frac{K}{S_0})-(r-\frac{1}{2}\sigma^2)T}{\sigma}}^{\infty} \frac{1}{\sqrt{2\pi T}} e^{-\frac{(x-T\sigma)^2}{2T}} dx.
\end{aligned}
$$

We denote

$$
d_1 := \frac{ln(\frac{S_0}{K}) + (r + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}. \tag{2.18}
$$

an by substitution of $y = \frac{x-T\sigma}{\sqrt{T}}$ we get

$$
\begin{aligned}
(I) &= S_0 \cdot \int_{-d_1}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy \\
&= S_0 \cdot \phi(d_1),
\end{aligned}
$$

where ϕ denotes the standard normal distribution function. Let us now compute the second part (II). We have

$$
(II) = e^{-rT} K \cdot \mathbb{P}^* \{ \frac{W_T^*}{\sqrt{T}} \geq \underbrace{\frac{ln(\frac{K}{S_0}) - (r - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}}_{-d_1+\sigma\sqrt{T}} \}
$$

Since $\frac{W_T^*}{\sqrt{T}} \sim \mathcal{N}(0,1)$ under $\mathbb{P}^*$, we get

$$
(II) = e^{-rT} K \cdot \phi(d_1 - \sigma\sqrt{T}).
$$

Hence we have proven the following theorem :

Theorem 2.7.1. Black-Scholes Formula *A fair price for a European call option with strike price K in the standard Black-Scholes model is given by*

$$\boxed{C(S_0, K, \sigma) = S_0 \cdot \phi(d_1) - e^{-rT} K \cdot \Phi(d_1 - \sigma\sqrt{T})}$$

where d_1 is given by the expression $d_1 := \frac{ln(\frac{S_0}{K}) + (r + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}$.

In the theorem above we do only speak of "a" fair price because it might be, that there are other fair prices corresponding to other equivalent martingale measures for the standard Black-Scholes model. In fact, there are no other equivalent martingale measures than the one (2.21), but we cannot (or don't want to) prove this at this place.

Exercise 2.7.1. *Compute a fair price for a digital call option with strike price K in the standard Black-Scholes model. The pay-out of a digital Call option is given as*

$$g(\omega) := 1_{\{X_T^1 \geq K\}}.$$

Exercise 2.7.2. *Compute a fair price for the Butterfly-Spread option with strike price K in the standard Black-Scholes model. The pay-out of a Butterfly-Spread option is given as*

$$g(\omega) = \begin{cases} 0 & if \quad X_T^1 \leq K \\ X_T^1 - K & if \quad K \leq X_T^1 \leq 2K \\ 2K - X_T^1 & if \quad 2K \leq X_T^1 \leq 3K \\ 0 & if \quad 3K \leq X_T^1 \end{cases} \tag{2.19}$$

Why is this option been called Butterfly-Spread ?

2.8 Why is the Black-Scholes model not good enough ?

The Black-Scholes model certainly is still the most famous and one of the most applied models. On the other side, it is known that the reality at financial markets looks different. In this short section we will demonstrate why this is so. The standard Black-Scholes model assumes for the stock price the price process

$$S_t = S_0 \cdot e^{(b-\frac{1}{2})t+\sigma W_t},$$

where the coefficients b and σ are assumed to be constants. σ is called the *volatility* of the stock. Let us consider the fair price of a European Call option given by Theorem 2.7.1 as

$$C(S_0, K, \sigma) = S_0 \cdot \phi(d_1) - e^{-rT} K \cdot \Phi(d_1 - \sigma\sqrt{T}), \tag{2.20}$$

Let us consider this price as a function of the volatility. From a rational point of view, it should be clear that this function is monotonically increasing. Nevertheless the ambitious reader should do the following exercise

Exercise 2.8.1. *Show the fair price of a European call option in the standard Black-Scholes model given by (2.24) is monotonically increasing as a function of the volatility σ.*

Hence we can look at the price at which European call options for this stock are traded on some stock exchange, call this price $C_{real}(S_0, K)$ and solve

$$C(S_0, K, \sigma) = C_{real}(S_0, K) \tag{2.21}$$

for σ. Using this volatility for the computation in (2.24) for exactly this option then of course gives the right (real world price). The fundamental problem though is, that when solving (2.27) in praxis, for different strike prices K, in general one gets different values for σ. So we have $\sigma = \sigma(K)$. This in fact contradicts the assumption made by the Black-Scholes model that σ is considered to be constant. $\sigma(K)$ is called the *implied volatility*, the graph of $\sigma(K)$ is often called *volatility smile* for its typical shape. There are various methods to improve the Black-Scholes model, and we will learn about some of them in the remaining of this lecture.

Chapter 3

Stochastic Integration

In section 2.3 we defined the sums $\sum_{\mathcal{Z}} d\varphi \tilde{X}^{tr}$ and $\sum_{\mathcal{Z}} \varphi d\tilde{X}^{tr}$ for certain partitions and assumed some kind of convergence of these sums when $|\mathcal{Z}| \to 0$. We now want to make this concept more precise. The concept is called stochastic Integration.

3.1 Semi-martingales

Let us first consider a generalization of the notion of a simple processes.

Definition 3.1.1. *An n-dimensional stochastic process $(X_t, \mathcal{F}_t)_{t\in[0,T]}$ is called* **simple predictable** *if there exist a finite sequence of stopping times*

$$0 = T_0 \leq T_1 \leq ... \leq T_m = T$$

with respect to the filtration $(\mathcal{F}_t)_{t\in[0,T]}$ and $\alpha_i : \Omega \to \mathbb{R}^n$ such that $|\alpha_i| < \infty$ a.s., α_i is $\mathcal{F}_{T_i}$ measurable and

$$X_t(\omega) = \alpha_0(\omega) \cdot 1_{\{0\}} + \sum_{i=1}^{m} \alpha_i(\omega) \cdot 1_{(T_{i-1},T_i]}, \ \forall t, \omega. \quad (3.1)$$

We denote the class of simple predictable processes with $\mathcal{E}$. We say a sequence of processes $X^k \in \mathcal{E}$ converges to $X \in \mathcal{E}$ if

$$X^k \xrightarrow{\mathcal{E}} X \quad :\Leftrightarrow \quad sup\{|X_t^k(\omega) - X_t(\omega)||(\omega,t) \in \Omega \times I\} \to 0 \textit{ as } k \to \infty$$

Definition 3.1.2. *A process $Y = (Y_t)_{t\in I}$ is called* **rcll** *(right continuous with left limits) if*

$$\begin{aligned} Y(\omega) &\in C_+(I,\mathbb{R}^n) \textit{ a.s.} \\ Y_{t-}(\omega) &:= \lim_{s\nearrow t} Y_s(\omega) \textit{ exists for a.a. } \omega \textit{ and for all } t \in I. \end{aligned}$$

In this case we define a new stochastic process as $Y_- := (Y_{t-})_{t\in I}$. A process $X = (X_t)_{t\in I}$ is called **lcrl** *(left continuous with right limits) if*

$$\begin{aligned} X(\omega) &\in C_-(I,\mathbb{R}^n) \textit{ a.s.} \\ X_{t+}(\omega) &:= \lim_{s\searrow t} X_s(\omega) \textit{ exists for a.a. } \omega \textit{ and for all } t \in I. \end{aligned}$$

and in this case as before we define $X_+ := (X_{t+})_{t\in I}$.We define the corresponding **jump processes** *as*

$$\Delta Y := Y - Y_- \quad , \quad \Delta X := X_+ - X.$$

We can now define, what a semi-martingale is.

Definition 3.1.3. *A process Y is called a* **total semi-martingale** *if Y is rcll, $(\mathcal{F}_t)_{t\in I}$ adapted and the map*

$$\begin{aligned} \int (\cdot) dY : \mathcal{E} &\to Map((\Omega,\mathcal{F}),(\mathbb{R},\mathcal{B}(\mathbb{R})) \\ X &\mapsto \alpha_0 Y_0 + \sum_{i=1}^n \alpha_i (Y_{T_{i+1}} - Y_{T_i}) \end{aligned}$$

where X is given by (3.1), is continuous in the following sense :

$$X^k \xrightarrow{\mathcal{E}} X \quad \Rightarrow \quad \int X^k dY \xrightarrow{\mathbb{P}} \int X dY \,.$$

Y is called a **semi-martingale** *if $\forall t \in I$ the stopped process Y^t is a total semi-martingale. We denote the class of semi-martingales with $\mathcal{S}$. In case $Y \in \mathcal{S}$ and $X \in \mathcal{E}$ we write*

$$\boxed{\int_0^t XdY := \int XdY^t.}$$

In the definition above Y was assumed to be $\mathbb{R}$-valued. Most definitions in this chapter correspond to the one-dimensional case but can be generalized to the n-dimensional case without taking serious effort. For example an n-dimensional semi-martingale is a $\mathbb{R}^n$ vector valued stochastic process whose components are semi-martingales.

Remark 3.1.1.

1. *$\mathcal{S}$ is a vector space.*
2. *$\int XdY$ is bilinear as a function of the integrand X and the integrator Y.*
3. *$\int_0^t XdY$ is $\mathcal{F}_t$ measurable and $(\int_0^t XdY)_{t\in I}$ is an $(\mathcal{F}_t)$ adapted rcll stochastic process.*
4. *$\mathcal{S} = \mathcal{S}(\mathbb{P})$ actually depends on $\mathbb{P}$, but if $\mathbb{Q} << \mathbb{P}$ then $Z^k \xrightarrow{\mathbb{P}} Z \Rightarrow Z^k \xrightarrow{\mathbb{Q}} Z$ hence every semi-martingale with respect to $\mathbb{P}$ is also a semi-martingale with respect to $\mathbb{Q}$. If in particular we have $\mathbb{P} \equiv \mathbb{Q}$ then $\mathcal{S}(\mathbb{P}) = \mathcal{S}(\mathbb{Q})$.*
5. *$\mathcal{S} = \mathcal{S}((\mathcal{F}_t))$ also depends on the filtration, but if $(\mathcal{G}_t)$ is a sub-filtration of $(\mathcal{F}_t)$ and Y is also $(\mathcal{G}_t)$-adapted, then Y is also a semi-martingale with respect to $(\mathcal{G}_t)$, since every elementary process with respect to $(\mathcal{G}_t)$ is also one with respect to $(\mathcal{F}_t)$.*

The following elementary properties of the stochastic integral (for elementary processes with respect to a semi-martingale) are left as an exercise.

Exercise 3.1.1. *Let $Y = (Y_t)_{t\in I}$ be a semi-martingale defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with given filtration $(\mathcal{F}_t)_{t\in I}$ and let X, X^1, X^2 be elementary processes. Show :*

1. $\int_0^t c dY = cY_t$ *for any* $c \in \mathbb{R}$ *considered as a constant deterministic process.*

2. $|Y_s| < \infty$ *a.s. for all* $s \in I$.

3. $Y \cdot 1_{\{Y \geq 0\}}$, $Y \cdot 1_{\{Y<0\}}$ *and* $|Y|$ *are semi-martingales.*

4. *If* $Y \geq 0$ *a.s. and* $X_s^1 \leq X_s^2$ *a.s.* $\forall\, s \in [0,t]$ *then* $\int_0^t X^1 dY \leq \int_0^t X^2 dY$

5. *If* X *is bounded and* Y *is a martingale, then* $(\int_0^t X dY)_{t\in I}$ *is also a martingale.*

The next propositions will give us various examples of semi-martingales.

Proposition 3.1.1. *Let* $Y = (Y_t)_{t\in I}$ *be an rcll* $(\mathcal{F}_t)_{t\in I}$ *adapted process s.t.* $\forall t$ *and* $\forall \omega$ *a.s. we have*

$$\int_0^t |dY_s|(\omega) := \sup\{\sum_{\mathcal{Z}(t)} |Y_{t_{i+1}} - Y_{t_i}| : \mathcal{Z}(t) : 0 = t_0 < t_1 ... < t_m = t\} < \infty$$

(we say Y *has finite variation on compacts). Then* $Y \in \mathcal{S}$.

Proof. Let $X \in \mathcal{E}$ be given as in (3.1). We have

$$\begin{aligned} |\int X dY^t| &= |\alpha_0 Y_0^t + \sum_{i=1}^n \alpha_i (Y_{T_{i+1}}^t - Y_{T_i}^t)| \\ &\leq |\alpha_0||Y_0^t| + \sum_{i=1}^n |\alpha_i||Y_{T_{i+1}}^t - Y_{T_i}^t| \\ &\leq \sup_{(\omega,t)\in\Omega\times I} |X_t(\omega)| \cdot (|Y_0| + \int_0^t |dY_s|) \end{aligned}$$

If $X^k \xrightarrow{\mathcal{E}} X$ then $X^k - X \xrightarrow{\mathcal{E}} 0$ and we follow from our Assumptions, that for any $\epsilon > 0$ we have $\lim_{n\to\infty} \mathbb{P}(|\int (X^k - X)dY^t| > \epsilon)$

$$\leq \lim_{n\to\infty} \mathbb{P}(\underbrace{\sup_{(\omega,t)\in\Omega\times I} |X_t^k(\omega) - X_t(\omega)|)|}_{\to 0 \text{ as } k\to 0} \cdot \underbrace{(|Y_0| + \int_0^t |dY^t|)}_{<\infty \text{ a.s.}} > \epsilon) = 0$$

□

The most interesting stochastic processes nevertheless do not have finite variation on compacts. So we have to go on. In the context of martingales we defined local martingales as something which becomes after appropriate stopping a martingale. The same procedure works in the semi martingale context but :

Proposition 3.1.2. *Let $Y = (Y_t)_{t\in I}$ be an rcll $(\mathcal{F}_t)_{t\in I}$ adapted process s.t. there exists an increasing sequence of stopping times T_n s.t. $limT_n = \infty$ a.s. and $\forall n$ we have $Y^{T_n} \in \mathcal{S}$, then $Y \in \mathcal{S}$ (in words : every local semi-martingale is a semi-martingale).*

Proof. Let $X \in \mathcal{E}$ be given as in (3.1). Define stopping times

$$S_n = T_n \cdot 1_{\{T_n\leq t\}} + \infty \cdot 1_{\{T_n > t\}}.$$

We have

$$\begin{aligned}\mathbb{P}(|\int XdY^t| \geq \epsilon) &\leq \mathbb{P}(|\int_0^t XdY| \geq \epsilon, S_n = \infty) + \mathbb{P}(|\int_0^t XdY| \geq \epsilon, S_n < \infty) \\ & \mathbb{P}(|\int_0^t XdY^{T_n\wedge t}| \geq \epsilon) + \mathbb{P}(S_n < \infty).\end{aligned}$$

Now let $X^k \xrightarrow{\mathcal{E}} X$ i.e. $X^k - X \xrightarrow{\mathcal{E}} 0$. Then $\forall n$

$$\lim_{k\to\infty} \mathbb{P}(|\int (X^k - X)dY^t| \geq \epsilon) \leq \underbrace{\lim_{k\to\infty} \mathbb{P}(|\int (X^k - X)dY^{T_n\wedge t}| \geq \epsilon)}_{=0} + \mathbb{P}(S_n < \infty)$$

The statement of the proposition now follows from $\lim_{n\to\infty} \mathbb{P}(S_n < \infty) = \lim_{n\to\infty} \mathbb{P}(T_n \leq t) = 0$.

□

Definition 3.1.4. *Let $Y = (Y_t)_{t\in I}$ be a martingale. We say Y is a*

1. *$L^2(\mathbb{P})$-martingale if $\forall t\ \mathbb{E}_{\mathbb{P}}(Y_t^2) = \int_\Omega Y_t^2 d\mathbb{P} < \infty$*
2. *$L^2(\mathbb{P}\otimes\mu)$-martingale if $\mathbb{E}_{\mathbb{P}\otimes\mu}(Y) = \int_{\Omega\times I} Y^2 d(\mathbb{P}\otimes\mu) < \infty$*

where μ denotes the Lebesgue-measure on $\mathbb{R}$. Further more, if Y is arbitrary, the we say Y is an

1. *$L^2_{loc}(\mathbb{P})$-martingale if there exists an increasing sequence of stopping times T_n s.t. $\lim T_n = \infty$ a.s. and Y^{T_n} is a $L^2(\mathbb{P})$-martingale*
2. *$L^2_{loc}(\mathbb{P}\otimes\mu)$-martingale if there exists an increasing sequence of stopping times T_n as above s.t. Y^{T_n} is a $L^2(\mathbb{P}\otimes\mu)$-martingale*

Exercise 3.1.2. *Let Y be an $L^2(\mathbb{P})$-martingale. Show that for any sequence of stopping times $0 = T_0 \leq T_1 \leq ... \leq T_{m+1} < \infty$ a.s. one has*

$$\mathbb{E}(\sum_{i=0}^{m}(Y_{T_{i+1}} - Y_{T_i})^2) = \mathbb{E}(Y^2_{T_{m+1}}).$$

The following proposition gives at once a large variety of examples for semi-martingales.

Proposition 3.1.3. *Let $Y = (Y_t)_{t\in I}$ be an rcll $L^2_{loc}(\mathbb{P})$-martingale. Then $Y \in \mathcal{S}$.*

Proof. W.l.o.g. we assume $Y_0 = 0$ and Y is an $L^2(\mathbb{P})$-martingale. Let $X \in \mathcal{E}$ as in (3.1). Then

$$\begin{aligned}
\mathbb{E}((\int_0^t X dY^t)^2) &= \mathbb{E}((\sum_{i=0}^{n}\alpha_i(Y^t_{T_{i+1}} - Y^t_{T_i}))^2) \\
&\leq (\sup_{(\omega,t)\in\Omega\times I} |X_t(\omega)|)^2 \mathbb{E}((Y^t_{T_{n+1}})^2) \\
&\leq (\sup_{(\omega,t)\in\Omega\times I} |X_t(\omega)|)^2 \mathbb{E}(Y_t^2)
\end{aligned}$$

where the last inequality follows from the Jensen-inequality (Prob. Theory Theorem 19.1 and Lemma 19.1). Hence

$X^k \xrightarrow{\mathcal{E}} X$ and hence $X^k - X \xrightarrow{\mathcal{E}} 0$ implies

$$\int X^k dY^t \xrightarrow{L^2(\mathbb{P})} \int X dY \ .$$

It follows from Prob. Theory Theorem 15.2c) that L^2 convergence implies convergence in probability. Hence we have

$$\int X^k dY^t \xrightarrow{\mathbb{P}} \int X dY$$

which proves the proposition.

□

Let us now consider our first explicit example of a semi-martingale :

Corollary 3.1.1. *Any Brownian motion $(W_t)_{t\in I}$ is a semi-martingale.*

Proof. Obviously (W_t) is an $L^2_{loc}(\mathbb{P})$ martingale. □

Proposition 3.1.4. *Let $Y = (Y_t)_{t\in I}$ be a continuous local martingale, then $Y \in \mathcal{S}$.*

Proof. Let T_n be an increasing sequence of stopping times s.t. $\lim_{n\to\infty} T_n = \infty$ a.s. and Y^{T_n} is a martingale. We define another increasing sequence of stopping times

$$S_n := inf\{t||Y_t| \geq n\}$$

Clearly $\lim_{n\to\infty} S_n = \infty$ a.s. and hence also $\lim_{n\to\infty} T_n \wedge S_n = \infty$ a.s.. $T_n \wedge S_n$ is also an increasing sequence of stopping times s.t. $Y^{T_n \wedge S_n} = (Y^{T_n})^{S_n}$ is a martingale. Furthermore we have

$$\mathbb{E}((\underbrace{Y^{T_n \wedge S_n}}_{\leq n})^2) \leq \mathbb{E}(n^2) = n^2 < \infty.$$

Hence Y_t is an $L^2_{loc}(\mathbb{P})$-martingale and by Proposition 3.1.3 this implies that Y is a semi-martingale.

□

Let us now describe what semi-martingales we've got so far. For this the following definition is useful.

Definition 3.1.5. *An rcll process $Y = (Y_t)_{t \in I}$ is called* **decomposable***, if*

$$\boxed{Y_t = Y_0 + M_t + A_t \; \forall \, t \in I}$$

where $M_0 = A_0 = 0$, M is an rcll $L^2_{loc}(\mathbb{P})$-martingale and A rcll and of finite variation on compacts.

Corollary 3.1.2. *Let $Y = (Y_t)_{t \in I}$ be a decomposable process, then Y is a semi-martingale.*

Proof. Follows from Proposition 3.1.1 and Proposition 3.1.3. □

The processes in mathematical finance are often of the form $Y_t = Y_0 + \tilde{M}_t + A_t$ where everything is as above with the exception that $\tilde{M}_t$ is assumed to be a local martingale. It follows from Proposition 3.1.4 that in this case Y is also a semi-martingale. In fact most semi-martingales are also decomposable but we won't go deeper into this.

3.2 The stochastic Integral

We already defined stochastic integration in case the integrator is a semi-martingale and the integrand is an elementary process. The class of integrators is already sufficient, but we want to enlarge the class of integrands. For this we exploit the continuity property in Definition 3.1.1. In fact we show that this continuity is actually stronger than it was assumed in Definition 3.1.1.

Definition 3.2.1. *Let X^k be a sequence of stochastic processes. We say X^k converges* **ucp** *(uniformly on compacts in probability) if*

$$\boxed{\forall t \in I: \ \sup_{0\le s\le t} |X_s^k - X_s| \xrightarrow{\mathbb{P}} 0 \quad :\Leftrightarrow \quad X^k \xrightarrow{ucp} X}$$

The following proposition shows that we can get any lcrl process as an ucp limit of elementary processes.

Proposition 3.2.1. *Let X be an lcrl adapted process. Then there exists a sequence X^k of elementary processes, such that $X^k \xrightarrow{ucp} X$.*

Proof. As before we define an increasing sequence of stopping times as

$$\sigma_n := \inf\{t : |X_t| \ge n\}.$$

We have $\lim_{n\to\infty} \sigma_n = \infty$ and hence

$$\begin{aligned}
\lim_{n\to\infty} \mathbb{P}(\sup_{0\le s\le t} |X_s^{\sigma_n} - X_s| \ge \epsilon) \ &\le \ \underbrace{\lim_{n\to\infty} \mathbb{P}(\sup_{0\le s\le \sigma_n} \overbrace{|X_s^{\sigma_n} - X_s|}^{=0} \ge \epsilon)}_{=0} \\
&+ \ \lim_{n\to\infty} \mathbb{P}(\sup_{\sigma_n\le s\le t} |X_s^{\sigma_n} - X_s| \ge \epsilon) \\
&\le \ \lim_{n\to\infty} \mathbb{P}(\sigma_n \le t) = 0
\end{aligned}$$

Hence $X^n \xrightarrow{ucp} X$ and all X^n are bounded processes. Hence we can assume in the following that X is also bounded. Define the rcll process Z as $Z = X_+$. For $\epsilon > 0$ define a sequence of stopping times

$$\tau_0^\epsilon := 0, \quad \tau_{n+1}^\epsilon := \inf\{t | t > \tau_n^\epsilon \text{ and } |Z_t - Z_{\tau_n^\epsilon}| > \epsilon\} \quad \forall\, n.$$

τ_n^ϵ is an increasing sequence of stopping times s.t. $\lim_{n\to\infty} \tau_n^\epsilon = \infty$ a.s. (here we need boundedness). We define

$$Z^\epsilon := \sum_{n=0}^{\infty} Z_{\tau_n^\epsilon} \cdot 1_{[\tau_n^\epsilon, \tau_{n+1}^\epsilon)}$$

Then we have that

$$|Z_t - Z_t^\epsilon| \leq \epsilon \text{ a.s. } \forall t \in I$$

and hence that $Z^\epsilon \to Z$ uniformly and bounded $\epsilon \to 0$. We define an lcrl process $\tilde{Z}^\epsilon$ as

$$\tilde{Z}^\epsilon := X_0 \cdot 1_{\{0\}} + \sum_{n=0}^{\infty} Z_{\tau_n^\epsilon} \cdot 1_{(\tau_n^\epsilon, \tau_{n+1}^\epsilon]}.$$

Then $\tilde{Z}^\epsilon \to Z_- = X$ uniformly and bounded as $\epsilon \to 0$. Now let

$$X^k := X_0 \cdot 1_{\{0\}} + \sum_{n=0}^{k} Z_{\tau_n^{1/k}} \cdot 1_{(\tau_n^{1/k} \wedge k, \tau_{n+1}^{1/k} \wedge k]}.$$

Clearly $X^k \in \mathcal{E}$ and for $k > t$ we have $X_s^k = \tilde{Z}_s^{1/k} \; \forall \, 0 \leq s \leq t$. Hence we get

$$\lim_{k\to\infty} \mathbb{P}(\sup_{0\leq s\leq t} |X_s^k - X_s| \geq \epsilon) \quad = \quad \lim_{k\to\infty} \mathbb{P}(\sup_{0\leq s\leq t} \underbrace{|\tilde{Z}_s^{1/k} - X_s|}_{\leq 1/k} \geq \epsilon) = 0$$

which shows that $X^k \xrightarrow{ucp} X$.

□

Proposition 3.2.2. *Let $Y = (Y_t)_{t\in I}$ be a semi-martingale and $X^k \in \mathcal{E}$ a sequence converging ucp to $X \in \mathcal{E}$, i.e. $X^k \xrightarrow{ucp} X$, then*

$$(\textstyle\int_0^t X^k dY)_{t\in I} \xrightarrow{ucp} (\int_0^t X dY \,)_{t\in I}.$$

Proof. W.l.o.g. we assume that Y is a total semi-martingale and $X = 0$. Let us first assume $X^k \in \mathcal{E}$ and $X^k \longrightarrow 0$ uniformly and bounded. let $\epsilon > 0$. We define stopping times

$$\tau_k := inf\{t : \int_0^t X^k dY| \geq \epsilon\}$$

Clearly $X^k \cdot 1_{[0,\tau_k]} \in \mathcal{E}$ and

$$\begin{aligned}
\mathbb{P}(\sup_{0\le s\le t}|\int_0^s X^k dY| \ge \epsilon) &= \mathbb{P}(\sup_{0\le s\le t}|\int_0^{\tau_k\wedge s} X^k dY| \ge \epsilon) \\
&= \mathbb{P}(\sup_{0\le s\le t}|\int_0^s X^k \cdot 1_{[0,\tau_k]} dY| \ge \epsilon) \\
&= \mathbb{P}(\sup_{0\le s\le t}|\int \underbrace{X^k \cdot 1_{[0,\tau_k]}}_{\xrightarrow{\mathcal{E}} 0} dY^s| \ge \epsilon) \to 0 \text{ as } k \to \infty.
\end{aligned}$$

This shows that $X^k \xrightarrow{\mathcal{E}} 0 \Rightarrow (\int_0^t X^k dY)_{t\in I} \xrightarrow{ucp} 0$. Now let $X^k \xrightarrow{ucp} 0$ with $X^k \in \mathcal{E}$. Let $\delta > 0, \epsilon > 0, t > 0$. It follows from above that there exists $\eta > 0$ s.t. whenever $\sup_{(\omega,t)\in\Omega\times I}|X_t(\omega)| \le \eta$ we have $\mathbb{P}(\sup_{0\le s\le t}|\int_0^s X dY| > \delta) < \epsilon/2$. Define stopping times

$$\sigma_k := \inf\{s : |X_s^k| > \eta\}.$$

We define

$$\tilde{X}_k := X^k \cdot 1_{[0,\sigma_k]} \cdot 1_{\{\sigma_k > 0\}}.$$

Then $\tilde{X}_k \in \mathcal{E}$ and $\sup_{(\omega,t)\in\Omega\times I}|\tilde{X}_t^k(\omega)| \le \eta$. If $\sigma_k \ge t$ then

$$\sup_{0\le s\le t}|\int_0^s \tilde{X}^k dY| = \sup_{0\le s\le t}|\int_0^s X^k dY|.$$

Hence we have

$$\begin{aligned}
\mathbb{P}(\sup_{0\le s\le t}|\int_0^s X^k dY| > \delta) &\le \mathbb{P}(\sup_{0\le s\le t}|\int_0^s \tilde{X}^k dY| > \delta) + \mathbb{P}(\sigma_k < t) \\
&\le \epsilon/2 + \underbrace{\mathbb{P}(\sup_{0\le s\le t}|X_s^k| > \eta)}_{\to 0 \text{ as } k\to\infty}
\end{aligned}$$

from which the statement of the proposition follows. □

We want to define the stochastic integral with integrand an arbitrary adapted lcrl process X as a limit of integrals of elementary processes. The following lemma shows that this limit indeed exists.

Lemma 3.2.1. *Let $X^n \in \mathcal{E}$ be a sequence of elementary processes s.t. $X^n \xrightarrow{ucp} X$ where X is lcrl adapted. Then $\forall t$ the sequence $\int_0^t X^n dY$ converges in probability.*

Proof. Applying Lemma 15.3 Prob. Theory, we have to show that $\int_0^t X^n dY$ is a Cauchy-sequence in probability. For this let $\epsilon > 0, \delta > 0$. For any $\eta > 0$ and $m, n \in \mathbb{N}$ we have

$$\begin{aligned}\mathbb{P}(|\int_0^t X^m dY - X^n dY| > \delta) &\leq \mathbb{P}(|\int_0^t (X^m - X^n) dY| > \delta, \sup_{0 \leq s \leq t} |X_s^m - X_s^n| \leq \eta) \\ &+ \mathbb{P}(\sup_{0 \leq s \leq t} |X_s^m - X_s^n| > \eta)\end{aligned}$$

Since $Y \in \mathcal{S}$ we can find $\eta > 0$ such that whenever $X \in \mathcal{E}$ with $\sup_{(\omega,t)\in\Omega\times I} |X_s(\omega)| \leq \eta$ we have $\mathbb{P}(|\int_0^t X dY| > \delta) < \epsilon/2)$. For this η we have

$$\mathbb{P}(|\int_0^t (X^m - X^n) dY| > \delta, \sup_{0 \leq s \leq t} |X_s^m - X_s^n| \leq \eta) < \epsilon/2.$$

Furthermore, since $X^n \xrightarrow{ucp} X \Rightarrow \sup_{0 \leq s \leq t} |X^n - X| \xrightarrow{\mathbb{P}} 0$ we can find $n_0 \in \mathbb{N}$ s.t.

$$\mathbb{P}(\sup_{0 \leq s \leq t} |X_s^n - X_s| > \eta/2) < \epsilon/4 \,\forall\, n \geq n_0.$$

Since

$$\{\sup_{0 \leq s \leq t} |X_s^n - X_s^m| > \eta\} \subset \{\sup_{0 \leq s \leq t} |X_s^n - X_s| > \eta/2\} \cup \{\sup_{0 \leq s \leq t} |X_s^m - X_s| > \eta/2\}$$

we have $\forall\, m, n \geq n_0$

$$\begin{aligned}\mathbb{P}(\sup_{0\le s\le t}|X_s^n - X_s^m| > \eta) &\le \mathbb{P}(\sup_{0\le s\le t}|X_s^n - X_s| > \eta/2) + \mathbb{P}(\sup_{0\le s\le t}|X_s^m - X_s| > \eta/2)\\ &< \epsilon/4 + \epsilon/4 = \epsilon/2.\end{aligned}$$

Hence $\forall\, m, n \ge n_0$ we have

$$\mathbb{P}(|\int_0^t X^m dY - \int_0^t X^n dY| > \delta) < \epsilon/2 + \epsilon/2 = \epsilon,$$

which shows that the sequence $\int_0^t X^n dY$ is indeed a Cauchy-sequence in probability.

□

We are now able to define the stochastic integral with integrand an adapted lcrl stochastic process.

Definition 3.2.2. *Let X be adapted and $lcrl$, $Y \in \mathcal{S}$. Then we define*

$$\int_0^t X dY := \mathbb{P} - \lim_k \int_0^t X^k dY$$

where (X^k) is an arbitrary sequence of elementary processes, s.t. $X^k \xrightarrow{ucp} X$ and $\mathbb{P} - \lim_k$ denotes the limit in probability. We consider $(\int_0^t X dY)_{t\in I}$ as a stochastic process and call it the stochastic integral of X with respect to Y. We denote this stochastic process as $\int X dY$.

The stochastic integral is well defined, since by Proposition 3.2.1 there exists at least one sequence of elementary processes converging ucp to X and by Proposition 3.2.2, if there are two sequences of elementary processes converging ucp to X, then the corresponding stochastic integrals converge ucp to the same stochastic process.From the definition and Remark 3.1.1 it is also clear that $\int X dY$ is adapted and rcll. The stochastic process $\int X dY$ should not be confused with the (image of the) map in 3.1.4.

As a first example we want to compute the stochastic integral $\int W dW$

where W is a Brownian-motion. For this we will make use of the following lemma.

Lemma 3.2.2. *Let $\mathcal{Z}_n : 0 = t_0^n < ... < t_{k_n}^n = t$ be a sequence of partitions such that* $\lim_{n\to\infty} |\mathcal{Z}_n| = 0$ *then*

$$\sum_i (W_{t_{i+1}^n} - W_{t_i^n})^2 \xrightarrow{\mathbb{P}} t\ .$$

Proof. Define $X^n := \sum_i (W_{t_{i+1}^n} - W_{t_i^n})^2$. Then by telescoping

$$X^n - t = \sum_i \underbrace{(W_{t_{i+1}^n} - W_{t_i^n})^2 - (t_{i+1}^n - t_i^n)}_{G_i}$$

The G_i's are independent random variables. Furthermore, with

$$\frac{W_{t_{i+1}^n} - W_{t_i^n}}{\sqrt{t_{i+1}^n - t_i^n}} \sim G \text{ with } G \sim \mathcal{N}(0,1).$$

we have

$$G_i \sim (G^2 - 1)(t_{i+1}^n - t_i^n).$$

$\mathbb{E}(G_i) = 0$ for all i and

$$\begin{aligned}
\mathbb{E}((X^n - t)^2) &= \mathbb{E}((\sum_i G_i)^2) = \sum_i \mathbb{E}(G_i^2) \\
&= \mathbb{E}((G^2-1)^2 \cdot \sum_i (t_{i+1}^n - t_i^n)^2) \\
&\leq \underbrace{\mathbb{E}((G^2-1)^2)}_{<\infty} \cdot \underbrace{|\mathcal{Z}_n|}_{\to 0} \cdot t \to 0 \text{ as } n \to \infty
\end{aligned}$$

This implies $X^n \xrightarrow{L^2(\mathbb{P})} t$ which then implies $X^n \xrightarrow{\mathbb{P}} t$.

□

Let us return to our example $\int W dW$. We know that $W \in \mathcal{S}$ and also that W is continuous, in particular lcrl. Hence this integral is well defined. Let us take a sequence of partitions $\mathcal{Z}_n : 0 = t_0^n < ... < t_{k_n}^n < \infty$ s.t. $\lim_{n\to\infty} |\mathcal{Z}_n| = 0$ and $\lim_{n\to\infty} t_{k_n}^n = \infty$. We define

$$X^n := \sum_i W_{t_i^n} \cdot 1_{(t_i^n, t_{i+1}^n]}$$

Exercise 3.2.1. *Show* $X^n \xrightarrow{ucp} W$.

We compute

$$\begin{aligned}
\int_0^t X^n dW &= \sum_i W_{t_i^n}(W_{t_{i+1}^n}^t - W_{t_i^n}^t) \\
&= \frac{1}{2}\sum_i \underbrace{(W_{t_{i+1}^n}^t + W_{t_i^n}^t)(W_{t_{i+1}^n}^t - W_{t_i^n}^t)}_{=(W_{t_{i+1}^n}^t)^2 - (W_{t_i^n}^t)^2} - \frac{1}{2}\sum_i (W_{t_{i+1}^n}^t - W_{t_i^n}^t)(W_{t_{i+1}^n}^t - W_{t_i^n}^t) \\
&= \frac{1}{2}\underbrace{W_{t_{k_n}^n \wedge t}^2}_{\to W_t^2} - \frac{1}{2}\underbrace{\sum_i (W_{t_{i+1}^n \wedge t} - W_{t_i^n \wedge t})^2}_{\to t \text{ (Lemma 3.2.2)}}
\end{aligned}$$

Building the limit for $n \to \infty$ we get

$$\boxed{\int_0^t W dW = \frac{1}{2}W_t^2 - \frac{1}{2}t. \qquad (3.2)}$$

This result is under some aspect very interesting and shows very good how stochastic integration is different from ordinary integration, where we have $\int x dx = \frac{1}{2}x^2$. The additional term $-\frac{1}{2}t$ comes from Lemma 3.2.2 and may surprise those, who prefer deterministic thinking. We will later see that from the probabilistic point of view, its existence is very natural. Let us now state (without proof) some properties of the stochastic integral.

Proposition 3.2.3. *Let X be an adapted lcrl process and $Y \in \mathcal{S}$. Then*

1. *For any stopping time τ we have $\int_0^\tau X dY = \int_0^\infty X \cdot 1_{[0,\tau]} dY = \int_0^\infty X dY_\tau$ where $\int_0^\infty$ means integration over the whole parameter set I.*

2. $(\Delta \int XdY)_s = X_s(\Delta Y)_s$, *hence if* Y *is a continuous semi-martingale, then* $\int XdY$ *is a continuous process.*

3. $\int XdY$ *depends on the measure* $\mathbb{P}$ *but if* $\mathbb{Q} << \mathbb{P}$ *then* $\mathbb{Q} - \int XdY$ *and* $\mathbb{P} - \int XdY$ *are* $\mathbb{Q}$ *indistinguishable.*

4. *Let* $\tilde{X}$ *be another adapted lcrl process and* $\tilde{Y}$ *be another semi-martingale. Let*

$$A = \{\omega | X(\omega) = \tilde{X}(\omega), Y(\omega) = \tilde{Y}(\omega)\}.$$

Then $\int XdY$ *and* $\int \tilde{X}d\tilde{Y}$ *coincide on* A.

5. *Associativity :* $\int XdY$ *is itself a semi-martingale and with* $\tilde{X}$ *as above we have*

$$\int \tilde{X} d \int XdY = \int \tilde{X}XdY.$$

The following proposition is a generalization of statement (5) in Exercise 3.1.1. in case Y is an $L^2_{loc}(\mathbb{P})$-martingale.

Proposition 3.2.4. *Let* Y *be an* $L^2_{loc}(\mathbb{P})$*-martingale and* X *adapted and lcrl. Then* $\int XdY$ *is also an* $L^2_{loc}(\mathbb{P})$*-martingale.*

Proof. W.l.o.g. we assume that Y is in fact an $L^2(\mathbb{P})$-martingale with $Y_0 = 0$ and X is bounded (the general case then follows via appropriate stopping as in the proofs before). Let $X^k \in \mathcal{E}$ be a sequence such that $X^k \xrightarrow{ucp} X$. W.l.o.g. this convergence is bounded by a constant c, i.e. $|X^k| \leq c$ for all k. Then (Exercise 3.1.1 (5)) $\int X^k dY$ is a martingale. Furthermore we have

$$\begin{aligned} \mathbb{E}((\int_0^t X^k dY)^2) &= \mathbb{E}(\sum_i X^k_{\tau^k_i} \cdot (Y^t_{\tau^k_{i+1}} - Y^t_{\tau^k_i}))^2 \\ &\leq c^2 \mathbb{E}((Y^t_{\tau^k_{n_k+1}})^2) \leq c^2 \mathbb{E}(\tau^2_t) \end{aligned}$$

where the last two inequalities are implied by Exercise 3.1.2 and the Jensen inequality. Hence the sequence $\int_0^t X^k dY$ is bounded in $L^2(\mathbb{P})$ and so is uniformly integrable. Hence by Exercise 1 Sheet 7 we have

$$\begin{aligned}
\mathbb{E}(\int_0^t X dY|\mathcal{F}_s) &= \lim_{k\to\infty} \mathbb{E}(\int_0^t X^k dY|\mathcal{F}_s) \\
&= \lim_{k\to\infty} \int_0^s X^k dY \\
&= \int_0^s X dY
\end{aligned}$$

□

Corollary 3.2.1. *If Y is a continuous local martingale and X is adapted and lcrl, then $\int X dY$ is a local martingale.*

Proof. In the proof of Proposition 3.1.4 we showed that any continuous local martingale is an $L^2_{loc}(\mathbb{P})$-martingale. □

Definition 3.2.3. *Let $\mathcal{Z}_n : 0 = \tau_0^n \leq \tau_1^n ... \leq \tau_{k_n}^n < \infty$ be a sequence of random partitions. We say $\mathcal{Z}_n$ tends to the identity if*

1. $\lim_n \sup_k \tau_k^n = \infty$ *a.s.*
2. $||\mathcal{Z}_n|| = \sup_k |\tau_{k+1}^n - \tau_k^n| \to 0$ *a.s. as* $n \to \infty$.

We say that $\mathcal{Z}_n$ is **refining**, *if $\mathcal{Z}_n \subset \mathcal{Z}_{n+1}$.*

The following Theorem (without proof) says that one can compute the stochastic integral by using random partitions tending to the identity.

Theorem 3.2.1. *Let $Y \in \mathcal{S}$ and X adapted and lcrl or rcll. Furthermore let $\mathcal{Z}_n$ be a sequence of random partitions as in Definition 3.2.3 tending to the identity. Then*

$$\sum_i X_{\tau_i^n} \cdot (Y^{\tau_{i+1}^n} - Y^{\tau_i^n}) \xrightarrow{ucp} \int X_- dY \ .$$

3.3 Quadratic Variation of a Semi-martingale

Definition 3.3.1. *Let* $X, Y \in \mathcal{S}$. *The* **quadratic covariation** *of* X *and* Y *is defined as*

$$[X,Y] \;:=\; XY - \int X_- dY - \int Y_- dX. \quad (3.3)$$

The **quadratic variation** *of* X *is defined as*

$$[X] \;:=\; [X,X] = X^2 - 2\int X_- dX. \quad (3.4)$$

Clearly $[X,Y]$ depends bilinearly and symmetrically on X and Y. Furthermore we have the **Polarization identity :**

$$[X,Y] = \frac{1}{2}([X+Y] - [X] - [Y]). \qquad (3.5)$$

The following proposition justifies the name “quadratic covariation“.

Proposition 3.3.1. *Let* $X, Y \in \mathcal{S}$. *Then the process* $[X,Y]$ *is adapted and of finite variation on compacts. If* $X = Y$ *then* $[X]$ *is increasing a.s. Furthermore for any refining sequence of partitions* $\mathcal{Z}_n : 0 = \tau_0^n \le \ldots \le \tau_{k_n}^n < \infty$ *tending to the identity we have*

$$X_0^2 + \textstyle\sum_i (X^{\tau_{i+1}^n} - X^{\tau_i^n})(Y^{\tau_{i+1}^n} - Y^{\tau_i^n}) \xrightarrow{ucp} [X,Y] \,. \qquad (3.7)$$

Proof. W.l.o.g. we assume $X_0 = 0$ and $X = Y$ (the general case follows from the polarization identity). We have

$$(X^2)^{\tau_{k_n}^n} \;=\; \textstyle\sum_{i=0}^{k_n-1} (X^2)^{\tau_{i+1}^n} - (X^2)^{\tau_i^n} \xrightarrow{ucp} X^2 \,. \qquad (3.8)$$

We know from Theorem 3.2.1 that

$$\sum_i X_{\tau_i^n}(X^{\tau_{i+1}^n} - X^{\tau_i^n}) \xrightarrow{ucp} \int X_- dY \tag{3.9}$$

Substratcting 2-times (3.9) from (3.8) under consideration of

$$(X^2)^{\tau_{i+1}^n} - (X^2)^{\tau_i^n} = (X^{\tau_{i+1}^n} + X^{\tau_i^n})(X^{\tau_{i+1}^n} - X^{\tau_i^n})$$

we get

$$\sum_i (X^{\tau_{i+1}^n} - X^{\tau_i^n})^2 \xrightarrow{ucp} X^2 - 2\int X_- dY = [X]\ .$$

In the sum above, any summand is positive and the greater the index t in $[X]_t$ the more summands are involved in the sum. Hence it is clear that the process $[X]$ is increasing a.s. and in particular of finite variation on compacts. That it is adapted follows immediately from the definition and the corresponding properties of the stochastic integral.

□

Proposition 3.3.2. *Under the assumptions of Proposition 3.3.1 we have*

1. $[X, Y] = X_0 Y_0$.
2. $\Delta[X, Y] = (\Delta X)(\Delta Y)$.
3. *For any stopping time* τ *we have* $[X^\tau, Y] = [X, Y^\tau] = [X^\tau, Y^\tau] = [X, Y]^\tau$.

Proof. (1.) and (3.) follow directly from the definition and the corresponding properties of the stochastic integral. To show (2.) we can assume w.l.o.g. that $X = Y$. From Proposition 3.2.3 (2) we have

$$\Delta \int X_- dX \quad = \quad X_- \Delta X.$$

Hence

$$\begin{aligned}(\Delta X)^2 &= (X - X_-)^2 \\ &= X^2 - 2XX_- + X_-^2 \\ &= X^2 - X_-^2 + 2X_-(X_- - X) \\ &= \Delta(X^2) - 2X_-\Delta X\end{aligned}$$

and

$$\Delta[X] = \Delta(X^2) - 2\Delta \int X_- dX = (\Delta X)^2.$$

□

As an immediate consequence of Definition 3.3.1 we get the following proposition :

Proposition 3.3.3. Integration by Parts Formula *Let* $X, Y \in \mathcal{S}$ *then* $X \cdot Y \in \mathcal{S}$ *and*

$$XY = \int X_- dY + \int Y_- dX + [X, Y]. \qquad (3.10)$$

We see that the classical Integration by Parts Formula from deterministic analysis is complemented by the quadratic covariation term of the two semi-martingales.

Definition 3.3.2. *For any rcll process* X *the continuous process* X^c *given as*

$$X_t^c = X_t - \sum_{0 \le s \le t} (\Delta X)_s \qquad (3.11)$$

is called the continuous part of X.

Exercise 3.3.1. *Under the assumptions of above, show X^c is a well defined continuous stochastic process.*

For any $X \in \mathcal{S}$ we have

$$
\begin{aligned}
[X]_t^c &= [X]_t - \sum_{0 \leq s \leq t} (\Delta[X])_s \\
&= [X]_t - \sum_{0 \leq s \leq t} (\Delta X)_s^2 \\
&= [X]_t - X_0^2 - \sum_{0 < s \leq t} (\Delta X)_s^2.
\end{aligned}
$$

Definition 3.3.3. *$X \in \mathcal{S}$ is called* **pure quadratic jump**, *if $[X]^c = 0$. In this case $[X]_t = X_0^2 + \sum_{0<s\leq t} (\Delta X)_s$.*

Exercise 3.3.2. *Show that any rcll process of finite variation on compacts is pure quadratic jump.*

Corollary 3.3.1. *Let $(W_t)_{t\in[0,\infty)}$ be a Brownian motion. Then (W_t) is of infinite variation on compacts. More generally any continuous semimartingale s.t. $[X]$ is not constant a.s. is of infinite variation on compacts.*

Proof. If X would be of finite variations on compacts, then by Exercise 3.3.2 and continuity we would have $[X] = X_0^2 = const.$ which would contradict the assumption. □

We have seen before, that $[W]_t = t$ and $W_t^2 - [W]_t = \int_0^t W dW$ follows a continuous local martingale.

Exercise 3.3.3. *Show by direct computation, that $W_t^2 - [W]_t = W_t^2 - t$ follows a martingale.*

This in fact is no coincidence, as the following proposition shows.

Lemma 3.3.1. *Let X be a continuous local martingale, then $X^2 - [X]$ is also a continuous local martingale. If $[X] \equiv 0$ then $X \equiv X_0$ is constant.*

Proof. By definition $X^2 - [X] = 2\int X_- dX$ and from Corollary 3.2.1 it follows that this is a local martingale. We follow from Proposition 3.2.3 that

$$\Delta \int X_- dX = X_- \Delta X \equiv 0.$$

Hence the local martingale $X^2 - [X]$ is also continuous. If $[X] \equiv 0$ then X^2 is a non-negative local martingale and hence a super martingale. Let us first assume that $X_0 = 0$. Then

$$\mathbb{E}(X_t^2) \leq \mathbb{E}(X_0^2) = 0$$

which shows that $X \equiv 0$. If $X_0 \neq 0$ then consider $\tilde{X} = X - X_0$ and repeat the argument.

□

Proposition 3.3.4. *Let X be a continuous local martingale and $\sigma \leq \tau$ be stopping times. If $[X]$ is constant on $[\sigma, \tau] \cap [0, \infty)$, then X is too.*

Proof. Consider the continuous local martingale $\tilde{X} := X^\tau - X^\sigma$. Using Proposition 3.3.2 we have

$$\begin{aligned} [\tilde{X}] &= [X^\tau - X^\sigma, X^\tau - X^\sigma] \\ &= [X^\tau, X^\tau] - 2 \underbrace{[X^\sigma, X^\tau]}_{=[X,X]^{\sigma\wedge\tau}=[X^\sigma]} + [X^\sigma, X^\sigma] \\ &= [X]^\tau - [X]^\sigma = 0, \end{aligned}$$

which together with Lemma 3.3.1 implies that $\tilde{X} \equiv 0$ and hence X constant on $[\sigma, \tau] \cap [0, \infty)$.

□

Proposition 3.3.5. *Let X, Y be $L^2_{loc}(\mathbb{P})$ martingales. Then $[X, Y]$ is the unique adapted rcll process A of finite variation on compacts, s.t.*

1. $XY - A$ *is a local martingale.*
2. $\Delta A = \Delta X \Delta Y$ *and* $A_0 = X_0 Y_0$.

Proof. It follows from Proposition 3.3.1 that $[X, Y]$ is of finite variation on compacts and adapted. Let us now show that $[X, Y]$ indeed satisfies conditions (1.) and (2.) above. By definition

$$[X, Y] = XY - \int X_- dY - \int Y_- dX,$$

and it follows from Proposition 3.2.4 that $XY - [X, Y]$ is a local martingale. Hence condition (1.) is satisfied. It follows from Proposition 3.3.2 that (2) is also satisfied. Now choose $A = [X, Y]$ and suppose B is another process satisfying the conditions from above. Then the process $A - B = (XY - B) - (XY - A)$ is a local martingale and

$$\Delta(A - B) = \Delta A - \Delta B = \Delta X \Delta Y - \Delta X \Delta Y = 0.$$

Hence $A - B$ is a continuous local martingale with $(A - B)_0 = A_0 - B_0 = 0$. Since $A - B$ is of finite variation on compacts it follows from Exercise 3.3.2 that $[A - B] = 0$.It then follows from Lemma 3.3.1 that $A - B \equiv 0$, hence $A = B$, which shows the uniqueness part in the proposition.

□

The proposition above can be quite useful in determining the process $[X, Y]$.

Exercise 3.3.4. *Let* X *be an* $L^2(\mathbb{P})$*-martingale, then* $\mathbb{E}(X_t^2) = \mathbb{E}([X]_t)$.

The next two propositions show how to compute the quadratic variation between two stochastic Integrals. We will omit the proofs. They can be found in the book [Protter].

Proposition 3.3.6. *Let* $Y_1, Y_2 \in \mathcal{S}$ *and* X_1, X_2 *be lcrl adapted processes. Then*

$$[\int X_1 dY_1, \int X_2 dY_2] \quad = \quad \int X_1 X_2 d[Y_1, Y_2]$$

Proof. [Protter], page 68. □

Proposition 3.3.7. *Let Z be rcll adapted and $X, Y \in \mathcal{S}$. Let $\mathcal{Z}_n : 0 \leq \tau_0^n \leq ... \leq \tau_{k_n}^n$ be a sequence of random partitions tending to the identity, then*

$$\sum_i Z_{\tau_i^n}(X^{\tau_{i+1}^n} - X^{\tau_i^n})(Y^{\tau_{i+1}^n} - Y^{\tau_i^n}) \xrightarrow{ucp} \int Z_- d[X, Y] \; .$$

Proof. [Protter], page 69. □

3.4 The Ito Formula

Let f be a sufficiently smooth function, and $Y \in \mathcal{S}$ be a semi-martingale. Is $f(Y)$ still a semi-martingale and if, how can we compute $f(Y)$ as in ordinary calculus, where we have the formula $f(y) = \int_0^y f'(x)dx$. An elementary example where this question arises is for example the standard Black-Scholes model, where we have

$$S_t = e^{(b - \frac{1}{2}\sigma^2)t + \sigma W_t}.$$

Is this a semi-martingale ? The Ito formula will completely answer all those questions. We will state it in a very general form, where it is also known as the Ito-Wentzel Formula, but only proof the case where the semi-martingale Y is continuous. This corresponds to the classical Ito Formula.

Theorem 3.4.1. Ito-Wentzel Formula *Let $Y \in \mathcal{S}$ and $f \in C^2(\mathbb{R})$ then $f(Y) \in \mathcal{S}$ and*

$$\begin{aligned} f(Y_t) &= f(Y_0) + \int_0^t f'(Y_-)dY + \frac{1}{2}\int_0^t f''(Y_-)d[Y]^c \\ &+ \sum_{0<s\leq t} (f(Y_s) - f(Y_{s-}) - f'(Y_{s-})(\Delta Y)_s). \end{aligned}$$

Proof. We assume Y is continuous, where we have to show that

$$f(Y_t) = f(Y_0) + \int_0^t f'(Y -_) \, dY + \frac{1}{2}\int_0^t f''(Y_-)d[Y].$$

let us consider the Taylor-expansion of f at the point $x \in \mathbb{R}$. We have

$$f(y) = f(x) + f'(x)(y-x) + \frac{1}{2}f''(x)(y-x)^2 + R(x,y),$$

where $R(x,y) \leq r(|y-x|) \cdot (y-x)^2$ for an increasing function $r : \mathbb{R}_+ \to \mathbb{R}_+$ with $\lim_{u\to 0} r(u) = 0$. W.l.o.g. $Y_0 = 0$ (compose f with a simple translation). We fix $t > 0$ and denote with $\mathcal{Z}_n$ a refining sequence of random partitions tending to the identity

$$\mathcal{Z}_n : 0 \leq \tau_0^n \leq ... \leq \tau_{k_n}^n = t \leq ... \leq \tau_{l_n}^n < \infty.$$

We have

$$\begin{aligned} f(Y_t) - f(Y_0) &= \sum_{i=0}^{k_n-1} (f(Y_{\tau_{i+1}^n}) - f(Y_{\tau_i^n}) \\ &= \sum_{i=0}^{k_n-1} f'(Y_{\tau_i^n})(Y_{\tau_{i+1}^n} - Y_{\tau_i^n}) \\ &+ \sum_{i=0}^{k_n-1} f''(Y_{\tau_i^n})(Y_{\tau_{i+1}^n} - Y_{\tau_i^n})^2 + \sum_{i=0}^{k_n-1} R(Y_{\tau_{i+1}^n}, Y_{\tau_i^n}). \end{aligned}$$

Using Theorem 3.2.1 and Proposition 3.3.7 we find that

$$\sum_{i=0}^{k_n-1} f'(Y_{\tau_i^n})(Y_{\tau_{i+1}^n} - Y_{\tau_i^n}) \xrightarrow{\mathbb{P}} \int_0^t f'(Y_-)dY$$

$$\sum_{i=0}^{k_n-1} f''(Y_{\tau_i^n})(Y_{\tau_{i+1}^n} - Y_{\tau_i^n})^2 \xrightarrow{\mathbb{P}} \frac{1}{2}\int_0^t f''(Y_-)d[Y]$$

Furthermore we have

$$|\sum_{i=0}^{k_n-1} R(Y_{\tau_{i+1}^n}, Y_{\tau_i^n})| \leq \sup_i r(|Y_{\tau_{i+1}^n} - Y_{\tau_i^n}|) \cdot \sum_{i=0}^{k_n-1}(Y_{\tau_{i+1}^n} - Y_{\tau_i^n})^2,$$

where the second expression on the right side converges in probability to $[Y]_t$. For each fixed ω the path $s \mapsto Y_s(\omega)$ is continuous and hence uniformly continuous on the compact interval $[0,t]$. Since $\lim_{n\to\infty}\sup_i |\tau_{i+1}^n - \tau_i^n| = 0$ this implies $\lim_{n\to\infty}\sup_i |Y_{\tau_{i+1}^n} - Y_{\tau_i^n}| = 0$ and hence using the properties of the function r

$$\lim_{n\to\infty}\sup_i r(|Y_{\tau_{i+1}^n} - Y_{\tau_i^n}|) = 0.$$

which implies that $|\sum_{i=0}^{k_n} R(Y_{\tau_{i+1}^n}, Y_{\tau_i^n})| \xrightarrow{\mathbb{P}} 0$.

□

There is also a multidimensional version of the Ito-formula. The proof is just a multidimensional modification of the previous proof, so we will omit it.

Theorem 3.4.2. *Let $Y = (Y^1, ..., Y^n)$ be an n-tuple of semi-martingales and $f \in C^2(\mathbb{R}^n, \mathbb{R})$, then $f(Y) \in \mathcal{S}$ and*

$$\begin{aligned} f(Y_t) &= f(Y_0) + \int_0^t \sum_{i=1}^n \frac{\partial}{\partial x^i} f(Y_-)dY^i + \frac{1}{2}\int_0^t \sum_{i,j=1}^n \frac{\partial^2}{\partial x^i \partial x^j} f(Y_-)d[Y^i, Y^j]^c \\ &+ \sum_{0<s\leq t}(f(Y_s) - f(Y_{s-}) - \sum_{i=1}^n \frac{\partial}{\partial x^i} f(Y_{s-})(\Delta Y)_s). \end{aligned}$$

As one can see, the transformation rules can be quite complicated since in general second order terms are involved. To circumvent this on can introduce the so called Fisk-Stratonovich-integral.

Definition 3.4.1. *Let $X, Y \in \mathcal{S}$. We define the* **Fisk-Stratonovich-integral** *of X with respect to Y as*

$$\int X \circ dY \quad := \quad \int X_- dY + \frac{1}{2}[X,Y]^c.$$

The Fisk-Stratonovich integral transform simpler than the Ito-integral as the following proposition shows.

Proposition 3.4.1. *Let $Y \in \mathcal{S}$ and $f \in C^3(\mathbb{R})$, then*

$$f(Y_t) = f(Y_0) + \int_0^t f'(Y_-) \circ dY + \sum_{0s \leq t} (f(Y_s) - f(Y_{s-})).$$

Exercise 3.4.1. *Prove Proposition 3.4.1.*

Of course there is also a multidimensional version of Proposition 3.4.1. Let us study two important examples on how the Ito-formula can be applied. The first one gives a characterization of Brownian motion, the second one is about the stochastic exponential, which in the framework of stochastic analysis is what the exponential function is in standard calculus.

Proposition 3.4.2. Lévy's characterization of Brownian motion : *Let $X = (X_t, \mathcal{F}_t)_{t\in[0,T]}$ be a stochastic process. Then the following two statements are equivalent :*

1. *X is a standard Brownian motion (on the interval $[0,T]$)*
2. *X is a continuous local martingale s.t. $[X]_t = t \; \forall t \in [0,T]$.*

Proof. □

Definition 3.4.2. *Let $X \in \mathcal{S}$. The stochastic exponential of X is defined as the stochastic process*

$$\mathcal{E}(X)_t := exp(X_t - \frac{1}{2}[X,X]^c_t) \prod_{0<s\leq t} (1+\Delta X_s)exp(-\Delta X_s).$$

Proposition 3.4.3. *Let $Z \in \mathcal{S}$ such that $Z > 0$ a.s. Then $X := \int \frac{1}{Z_-} dZ$ is well defined and the unique semi-martingale which satisfies $Z = Z_0 \cdot \mathcal{E}(X)$.*

Proof. We only give a proof for the case when Z is continuous. We apply the Ito formula to the function

$$\begin{aligned} f : \mathbb{R}^2 &\to \mathbb{R} \\ (x,y) &\mapsto e^{x-\frac{1}{2}y} \end{aligned}$$

which obviously satisfies $\mathcal{E}(X)_t = f(X_t, [X]_t)$. Hence we get

$$\begin{aligned} d\mathcal{E}(X) &= \mathcal{E}(X)dX - \frac{1}{2}\mathcal{E}(X)d[X] + \frac{1}{2}\mathcal{E}(X)d[X] \\ &= \mathcal{E}(X)dX. \end{aligned}$$

We also have $dX = \frac{1}{[}Z]dZ$ which is equivalent to $dZ = ZdX$. The Integration by parts formula yields :

$$d(\frac{Z}{\mathcal{E}(X)}) = Zd(\frac{1}{\mathcal{E}(X)}) + \frac{1}{\mathcal{E}(X)}dZ + d[Z, \frac{1}{\mathcal{E}(X)}].$$

Applying the Ito-formula to $\frac{1}{\mathcal{E}(X)} = f(-X, -[X])$ yields :

$$\begin{aligned}
d\left(\frac{1}{\mathcal{E}(X)}\right) &= -(\mathcal{E}(X))^{-1}dX + \frac{1}{2}\mathcal{E}(X)^{-1}d[X] + \frac{1}{2}\mathcal{E}(X)^{-1}d[X] \\
&= \mathcal{E}(X)^{-1}(-dX + d[X]). \\
\Rightarrow d\left(\frac{Z}{\mathcal{E}(X)}\right) &= \mathcal{E}(X)^{-1}(-ZdX + Zd[X] + \underbrace{dZ}_{=ZdX} - \underbrace{d[Z,X]}_{=Zd[X]} = 0.
\end{aligned}$$

hence $Z/\mathcal{E}(X) \equiv const.$ and since $\mathcal{E}(X)_0 = 1$ this constant must be Z_0. If $Y \in \mathcal{S}$. So we have proven $Z = Z_0 \cdot \mathcal{E}(X)$. If $Y \in \mathcal{S}$ would be another semi-martingale which satisfies $Z = Z_0 \cdot \mathcal{E}(Y)$ then by the same argumentation as for X we have

$$dY = \frac{1}{\mathcal{E}(Y)}d\mathcal{E}(Y) = \frac{1}{Z}dZ = dX$$

which shows $X = Y$. Hence X is uniquely determined by this property.

□

Remark 3.4.1. *An immediate consequence of Proposition 3.4.3 is that for any $X \in \mathcal{S}$ we have $d\mathcal{E}(X) = \mathcal{E}(X)_{-}dX$.*

3.5 The Girsanov Theorem

Assume you have been given two equivalent measures $\mathbb{P} \sim \mathbb{Q}$ and a semi-martingale $X = M + A$ decomposed into a local martingale (under $\mathbb{P}$) part and a process A which has finite variation on compacts. In general M is no local martingale under $\mathbb{Q}$. Nevertheless the Girsanov theorem says, there is a decomposition of $X = N + C$ into a local martingale (under $\mathbb{Q}$) part and a process C of finite variation on compacts and it tells you precisely how to compute N and C. We will see that this theorem is the main tool in the computation of equivalent martingale measures for financial markets.

Theorem 3.5.1. (Girsanov) *Let $\mathbb{P} \equiv \mathbb{Q}$ equivalent probability measures and let $Z_s = \mathbb{E}_{\mathbb{P}}(\frac{d\mathbb{Q}}{d\mathbb{P}}|\mathcal{F}_s)$ denote the density process. Furthermore let $X = M + A$ where M is a local martingale under $\mathbb{P}$ and A is of finite variations on compacts. Then the process N defined as*

$$N_t = M_t - \int_0^t \frac{1}{Z_{s-}} d[Z, M]_s - \sum_{0 \le s \le t} \Delta(\frac{1}{Z})_s(\Delta Z)_s)(\Delta M)_s$$

is a local martingale under $\mathbb{Q}$, $C := X - N$ is of finite variation on compacts and $X = N + C$.

Before we can proof this, wee need two additional lemmas.

Lemma 3.5.1. *If $Y \in \mathcal{S}$ and X is rcll of finite variation on compacts. Then*

$$[X, Y] = X_0 Y_0 + \sum_{0 < s \le t} (\Delta X)_s (\Delta Y)_s.$$

Proof. If Y is continuous for any sequence $\mathcal{Z}_n : 0 = \tau_0^n \le ... \le \tau_{k_n}^n < \infty$ of random partitions tending to the identity, we have

$$|\sum_i (X^{\tau_{i+1}^n} - X^{\tau_i^n})(Y^{\tau_{i+1}^n} - Y^{\tau_i^n}) \le \underbrace{\sup_i |Y^{\tau_{i+1}^n} - Y^{\tau_i^n}|}_{\to 0 \text{ continuity !}} \cdot \underbrace{\int |dX|}_{<\infty} \to 0$$

Hence it follows from Proposition 3.3.1 that $[X, Y]^c = [X, Y] - X_0 Y_0 = 0$. In case Y is not continuous we show that $[X, Y]^c = [X, Y^c]$. Then we can conclude from above that $[X, Y]^c = [X, Y^c]^c = 0$. We have

$$\sum_i (X^{\tau_{i+1}^n} - X^{\tau_i^n})(Y^{c,\tau_{i+1}^n} - Y^{c,\tau_i^n}) \quad \to \quad [X, Y^c]. \tag{3.12}$$

We can assume that between τ_i^n and τ_{i+1}^n X and Y jump at most one time. Then

$$\begin{aligned}
\sum_i (X^{\tau_{i+1}^n} - X^{\tau_i^n})(Y^{c,\tau_{i+1}^n} - Y^{c,\tau_i^n}) &= \sum_i (X^{\tau_{i+1}^n} - X^{\tau_i^n})(Y^{\tau_{i+1}^n} - Y^{\tau_i^n} - \sum_{\tau_i^n < s \leq \tau_{i+1}^n} (\Delta Y)_s) \\
&= \sum_i (X^{\tau_{i+1}^n} - X^{\tau_i^n})(Y^{\tau_{i+1}^n} - Y^{\tau_i^n} \\
&- \underbrace{\sum_i \underbrace{(X^{\tau_{i+1}^n} - X^{\tau_i^n})}_{\to \sum_{\tau_i^n < s \leq \tau_{i+1}^n} (\Delta X)_s)} \sum_{\tau_i^n < s \leq \tau_{i+1}^n} (\Delta Y)_s)}_{\to \sum_{0 < s \leq t} (\Delta X)_s (\Delta Y)_s}
\end{aligned}$$

which shows that the expression on the left side in (3.12) also converges to $[X, Y]^c$ and hence $[X, Y^c] = [X, Y]^c$.

□

Lemma 3.5.2. *Let $\mathbb{Q} \sim \mathbb{P}$ be two equivalent measures and let $\mathbb{Z}_t = \mathbb{E}_{\mathbb{P}}(\frac{d\mathbb{Q}}{d\mathbb{P}}$ be the corresponding density process. Then the following two statements are equivalent :*

1. *M is a local martingale under $\mathbb{Q}$.*
2. *$M \cdot Z$ is a local martingale under $\mathbb{P}$.*

Proof. For a localizing sequence τ_n of stopping times, we have $\lim_n \tau_n = 0\, \mathbb{P} a.s. \Leftrightarrow \lim_n \tau_n = 0\, \mathbb{Q} a.s.$ Hence after stopping we can restrict ourselves to martingales. Let $0 \leq s \leq t$ and $A \in \mathcal{F}_s$. Then $\forall r \geq s$

$$\int_A M_r d\mathbb{Q} = \int_A M_r \frac{d\mathbb{Q}}{d\mathbb{P}} d\mathbb{Q} = \int_A M_r Z_r d\mathbb{P}.$$

Hence we have

$$\begin{aligned}
\int_A M_t d\mathbb{Q} &= \int_A M_s d\mathbb{Q} \\
\Leftrightarrow \int_A M_t Z_t d\mathbb{P} &= \int_A M_s Z_s d\mathbb{P}
\end{aligned}$$

which implies

$$\mathbb{E}_{\mathbb{Q}}(M_t|\mathcal{F}_s) = M_s \Leftrightarrow \mathbb{E}_{\mathbb{P}}(M_t Z_t|\mathcal{F}_s) = M_s Z_s.$$

□

Proof. (of Theorem 3.5.1) Z,M and by Proposition 3.3.3 also

$$Z \cdot M - [Z, M] = \int Z_- dM + \int M_- dZ \tag{3.13}$$

are local martingales under $\mathbb{P}$. It follows from

$$\frac{1}{Z_s} = \mathbb{E}_{\mathbb{Q}}(\frac{\mathbb{P}}{\mathbb{Q}}|\mathcal{F}_s)$$

that $\frac{1}{Z_s}$ is a local martingale under $\mathbb{Q}$. It follows from Lemma 3.5.2 and (3.13) that

$$M - \frac{1}{Z}[Z, M] = \frac{1}{Z}(\int Z_- dM + \int M_- dZ) \tag{3.14}$$

is a local martingale under $\mathbb{Q}$. Integration by parts (under $\mathbb{Q}$) yields

$$\frac{1}{Z}[Z, M] = \int \frac{1}{Z_-} d[Z, M] + \int [Z, M] d(\frac{1}{Z}) + [[Z, M], \frac{1}{Z}].$$

let us define

$$\tilde{N} := \int [Z, M] d(\frac{1}{Z}). \tag{3.15}$$

We have

$$\begin{aligned} \frac{1}{Z_t}[Z, M]_t &= \int \frac{1}{Z_{s-}} d[Z, M]_s + \tilde{N}_t + [\overbrace{[Z, M]}^{FV}, \frac{1}{Z}]_t \\ &\underbrace{=}_{\text{Lemma 3.5.1}} \int \frac{1}{Z_{s-}} d[Z, M]_s + \tilde{N}_t + \sum_{0 \le s \le t} \Delta(\frac{1}{Z})_s \underbrace{\Delta[Z, M]_s}_{(\Delta Z)_s)(\Delta M)_s} . \end{aligned}$$

Since the sum of two local martingales is a gain a local martingale we get by adding (3.15) and (3.14) a local martingale under $\mathbb{Q}$

$$\begin{aligned} & \tilde{N}_t + M_t - \int_0^t \frac{1}{Z_{s-}} d[Z,M]_s - \tilde{N}_t - \sum_{0 \le s \le t} \Delta(\frac{1}{Z})_s (\Delta Z)_s)(\Delta M)_s \\ = \; & M_t - \int_0^t \frac{1}{Z_{s-}} d[Z,M]_s - \sum_{0 \le s \le t} \Delta(\frac{1}{Z})_s (\Delta Z)_s)(\Delta M)_s \end{aligned}$$

but this is exactly the process N defined in the Theorem. Furthermore

$$C_t = X_t - N_t = A_t + M_t - N_t = A + \int \frac{1}{Z_{s-}} d[Z,M]_s + \sum_{0 \le s \le t} \Delta(\frac{1}{Z})_s (\Delta Z)_s)(\Delta M)_s$$

is of finite variation on compacts. This finishes the proof.

□

Remark 3.5.1. *If in the setting of the Girsanov Theorem either the local martingale part M under $\mathbb{P}$ or the density process Z is continuous, then*

$$N = M - [\tilde{Z}, N]$$

where $\tilde{Z} = \int \frac{1}{Z_-} dZ$ satisfies $\mathcal{E}(\tilde{Z}) = Z$.

The Girsanov Theorem tells you, how you can decompose a semimartingale in its local martingale part and its finite variation part after a change of measure. The following proposition shows how to apply the Girsanov Theorem effectively in the Brownian motion setting. In fact this proposition is also sometimes called the Girsanov Theorem.

Proposition 3.5.1. *Let $\mathbb{W}$ be a d-dimensional Brownian motion on $(\Omega, \mathcal{F}, \mathbb{P})$ with respect to the filtration $(\mathcal{F}_t)$. Let H be a lcrl adapted d-dimensional process such that the local martingale $\mathcal{E}(-\int HdW)$ is in fact a martingale, then the process defined by*

$$X_t := W_t + \int_0^t H_s ds$$

is a Brownian motion on $[0,T]$ *with respect to the probability measure* $\mathbb{Q}$ *defined by* $\mathbb{Q}(A) = \mathbb{E}_{\mathbb{P}}(1_A \cdot Z_T)$ *where* $Z_T := \mathcal{E}(-\int HdW)_T$.

Proof. W.l.o.g. we assume $d = 1$. It follows from Remark 3.4.1 that $dZ_t = Z_t d(-\int_0^t H_s dW_s) = -Z_t H_t dW_t$. Clearly $\frac{d\mathbb{Q}}{d\mathbb{P}} = Z_T$ and since Z_T is a martingale, we also have $\mathbb{E}_{\mathbb{P}}(\frac{d\mathbb{Q}}{d\mathbb{P}}|\mathcal{F}_t) = Z_t$. Hence we are in the framework of the Girsanov Theorem and it follows that

$$N_t := W_t - \int_0^t \frac{1}{Z_s} d[Z,W]_s$$

is a continuous local martingale under $\mathbb{Q}$. Furthermore by Proposition 3.3.6 we have

$$[Z,W]_t = \int_0^t -Z_s H_s d[W,W]_s = -\int_0^t Z_s H_s ds.$$

This implies that $X_t = W_t + \int_0^t H_s ds = N_t$ is a continuous local martingale under $\mathbb{Q}$. Since $\int_0^t H_s ds$ is of finite variation on compacts we have $[X]_t = [W]_t = t$ and the proposition follows from the Lévy characterization of Brownian motion. □

3.6 The Stochastic Integral for predictable Processes

In this section we will enlarge the set of integrands. So far we are able to integrate lcrl processes with respects to semi-martingales. The space of lcrl processes seem to be large but is still not large enough. In the economic setting we think of the space of integrands as the space of trading strategies. We wish that our financial market is complete (at least the Black-Scholes Versions). To achieve this, lcrl processes are unfortunately not enough. In this section we sketch how to extend the stochastic integral to predictable processes. We will make use of the following Theorem which translates the term semi-martingale into the more familiar terms of local martingales and finite variation processes.

Theorem 3.6.1. Characterization of Semi-martingales *Let X be an adapted rcll process. Then the following things are are equivalent :*

1. $X \in \mathcal{S}$.
2. *X is decomposable (see Definition 3.1.5).*
3. *Given $\beta > 0$ there exists a local martingale M such that $|\delta M| \leq \beta$ a.s. and $M_0 = 0$, a process A which is of finite variation on compacts s.t. $A_0 = 0$ and $X_t = X_0 + M_t * A_t$.*

The decomposition in (3) is in fact unique, if one assumes one extra condition on the process A which is called naturality.

Definition 3.6.1. *For a semi-martingale Y decomposed as in (3) of Theorem 3.7.1 in $Y_t = X_0 + M_t + A_t$ we define*

$$||Y||_{\mathcal{H}^2} = (\mathbb{E}([M]_\infty))^{1/2} + (\mathbb{E}(\int_0^\infty |dA_s|^2))^{1/2}.$$

where $[M]_\infty = \lim_{t\to\infty}[M]_t$ and $|dA_s|$ is the total variation of A as defined in Proposition 3.1.1. We define the space

$$\mathcal{H}^2 := \{Y \in \mathcal{S} : ||Y||_{\mathcal{H}^2} < \infty\}.$$

Definition 3.6.2. *The σ-algebra*

$$\mathcal{P} := \sigma(X | X : \Omega \times I \to \mathbb{R} \textit{ lcrl adapted })$$

is called the **predictable** *σ-algebra on $\Omega \times I$. A process X is called predictable if it is $\mathcal{P}/\mathcal{B}(\mathbb{R})$ measurable.*

Clearly any lcrl process is predictable. In the following Definition we define a metric on the set of all predictable processes. This metric depends on the semi-martingale Y, which serves as the integrator.

Definition 3.6.3. *Let $Y \in \mathcal{H}^2$ and $X, \tilde{X}$ predictable processes. We define*

$$d_Y(X, \tilde{X}) := (\mathbb{E}(\int_0^\infty (X_s - \tilde{X}_s) d[M]_s))^{1/2} + (\mathbb{E}(\int_0^\infty |X_s - \tilde{X}_s||dA_s|^2))^{1/2}.$$

The integrals in this definition are path-wise Riemann-Stieltjes integrals.

With these definitions we are now able to generalize the stochastic integral to the case of predictable integrands.

Theorem 3.6.2. *Let $Y \in \mathcal{H}^2$ and X a predictable process with paths that are bounded on compact intervals. Then there exists a sequence X^n of lcrl processes such that $X^n \xrightarrow{d_Y} X$ and $\lim_{n\to\infty} X^n dY$ exists in $\mathcal{H}^2$. We define $\int X dY$ as this limit.*

The following is called the Ito-isometry.

Proposition 3.6.1. *Let $Y \in \mathcal{H}^2$ a local martingale and X a predictable process with paths that are bounded on compact intervals. Then*

$$\mathbb{E}((\int_0^\infty X dY)^2) = ||\int_0^\infty X dY||_{\mathcal{H}^2} = \mathbb{E}(\int_0^\infty X_t^2 d[Y]_t).$$

Proof. After stopping we can assume that $\int X dY$ is a martingale. It then follows from Exercise .. that

$$\begin{aligned} \mathbb{E}((\int_0^\infty X dY)^2) &= \mathbb{E}(\lim_t [\int X dY]_t) \\ &= \mathbb{E}(\lim_t \int_0^t X_s d[Y]_s) \\ &= \mathbb{E}(\int_0^\infty X_s d[Y]_s) \end{aligned}$$

□

Taking $Y = W$ a Brownian motion we get the following Corollary :

Corollary 3.6.1. $\mathbb{E}((\int_0^T X_s dW_s)^2) = \mathbb{E}(\int_0^t X_s^2 ds)$.

3.7 The Martingale Representation Theorem

In this section we present and prove the martingale representation theorem, which has many application, but for us its relation to the question of completeness of financial markets is most important. We will study this relation in the next chapter.

Theorem 3.7.1. *Let $\mathbb{W}$ be a n-dimensional Brownian motion and $(\mathcal{F}_t) = (\overline{\mathcal{F}}_{t+}^B)$ be the augmented, right continuous Brownian Filtration. Furthermore let Y be a m-dimensional rcll local martingale with respect to the filtration $(\mathcal{F}_t)$. Then there exists a predictable $(m \times n)$-matrix valued process X such that*

$$\boxed{Y = \int X d\mathbb{W}.}$$

Chapter 4

Explicit Financial Market Models

In this chapter we give explicit models for financial markets. Of course we can only present some of the large variety of financial models, but we try to give at least some example for any class of market model, such there is generalized Black Scholes, diffusion type stochastic volatility and markets with jump components. Throughout the whole chapter the triple $(\Omega, \mathcal{F}, \mathbb{P})$ stands for a complete probability space.

4.1 The generalized Black Scholes Model

This model is a generalization of the model presented in Chapter 1 in the sense that it does not assume that the trend and the volatility are given constants, but stochastic processes which are adapted to the underlying filtration. For the underlying filtration we take $\mathcal{F}_t = \overline{\mathcal{F}}_{t+}^{\mathbb{W}}$ the augmented and right-continuous Brownian filtration which is generated by a m-dimensional Brownian motion $\mathbb{W}$ on $(\Omega, \mathcal{F}, \mathbb{P})$. The price process $X = (X^0, X^1, ..., X^n)$ which contains only tradeable assets and is modeled over the finite interval $[0, T]$ is given as

$$
\begin{aligned}
X_t^0 &= exp(\int_0^t r_s ds) \\
X_t^i &= X_0^i exp(\int_0^t (b_s^i - \frac{1}{2}\sum_{j=1}^m \sigma_{ij,s}^2)ds + \sum_{j=1}^m \sigma_{ij,s}^2 dW_s^j)\ , \forall i \in \{1,..,n\}
\end{aligned}
$$

Here (r_t), (b_t) and (σ_t) are scalar resp. vector resp. matrix valued predictable (with respect to the filtration $(\mathcal{F})_t$) stochastic processes which are pathwise bounded. We do further assume that there exists a constant $K > 0$ such that $||\sigma_t x|| \geq K||x||$ for all $x \in \mathbb{R}^m$. This property is often called uniform coercitivity. using the Ito-formula we can write the price-processes in differential form :

$$
\begin{aligned}
dX_t^0 &= X_t^0 r_t dt \\
dX_t^i &= X_0^i \cdot (b_t^i dt + \sum_{j=1}^m \sigma_{ij,t}^2 dW_t^j)\ , \forall i \in \{1,..,n\}.
\end{aligned}
$$

Application of the Integration by parts formula on $\tilde{X}_t^i = X_t^i \cdot (X_t^0)^{-1}$ gives

$$
\begin{aligned}
d\tilde{X}_t^0 &= 0 \\
d\tilde{X}_t^i &= \tilde{X}_0^i \cdot ((b_t^i - r_t) + \sum_{j=1}^m \sigma_{ij,t}^2 dW_t^j)\ , \forall i \in \{1,..,n\}.
\end{aligned}
$$

For the set of trading strategies Φ we assume the following : $\varphi = (\varphi^0, \varphi^1, ..., \varphi^n) \in \Phi$ if it is a predictable $\mathbb{R}^{n+1}$-valued stochastic process process such that

1. $\int_0^T |\varphi_t^0| dt < \infty$ and $\int_0^T (\varphi_t^i \cdot X_t^i)^2 dt < \infty$ a.s.
2. $V_t(\phi) = \varphi_t \cdot X_t = \int_0^t \sum_{i=1}^n \varphi_t^i dX_t^i \Leftrightarrow d(\varphi \cdot \tilde{X}) = \varphi \cdot d\tilde{X}$ “self-financing“
3. The price-process $V_t(\varphi)$ is bounded from below. “tameness “

The equivalence in two follows from application of the Integration by parts formula. We denote the corresponding financial market as

$$\mathcal{M}_{genBS} = ((X_t, \mathcal{F}_t)_{t\in[0,T]}, \Phi)$$

and call it the **generalized Black-Scholes financial market**. Let us now compute the equivalent martingale measures for this financial market. Assume that $\mathbb{P}^* \in \mathcal{P}(\mathcal{M}_{genBS})$ and let $Z_t := \mathbb{E}_{\mathbb{P}}(\frac{d\mathbb{P}^*}{d\mathbb{P}}|\mathcal{F}_t)$ denote the corresponding density process. Clearly Z is a strictly positive martingale with respect to the Brownian filtration, hence by Theorem 3.7.1 continuous. $\tilde{X}^0 \equiv 1$ is a martingale in any case but for $\tilde{X}^i$ for $1 \le i \le n$ to be a local martingale under $\mathbb{P}^*$, it follows from the Girsanov theorem (Theorem 3.5.1 and Remark 3.5.1) that we must have

$$\tilde{X}^i - \tilde{X}^0 = M^i - [\tilde{Z}, M^i] \tag{4.1}$$

where $\tilde{X}^i = \tilde{X}^i_0 + M^i + A^i$ is the decomposition of X into initial value, local martingale part and finite variation part under the measure $\mathbb{P}$ (see Theorem 3.6.1 (3)) and $\tilde{Z} = \int \frac{1}{Z} dZ$. Since $\tilde{Z}$ is also a local martingale with respect to the Brownian filtration, by Theorem 3.7.1 (Martingale Representation Theorem) there must be an m-dimensional predictable process θ such that

$$d\tilde{Z} = \theta \cdot d\mathbb{W} = \sum_{j=1}^{m} \theta^j dW^j. \tag{4.2}$$

If we use this form of $\tilde{Z}$ to compute the covariation process in (4.1) we get

$$d[\tilde{Z}, M^i]_t = \tilde{X}^i_t \sum_{j=1}^{m} \sigma_{ij,t} \theta^j_t dt \tag{4.3}$$

which we can use to compute $d\tilde{X}^i$ as

$$d\tilde{X}_t^i = \tilde{X}_t^i(\sum_{j=1}^{m} \sigma_{ij,t}\theta_t^j dW_t^j - \sum_{j=1}^{m} \sigma_{ij,t}\theta_t^j dt). \tag{4.4}$$

Since the decomposition into initial value, local martingale part and finite variation part is unique by comparison with the expression for $d\tilde{X}^i$ on the previous page we must have $b_t^i - r_t = -\sum_{j=1}^m sigma_{ij,t}\theta_t^j$ for $1 \leq i \leq n$. Using vector notation we can write this as

$$\boxed{\sigma_t \cdot \theta_t = \underline{r}_t - b_t \text{ a.s } \forall t \in [0,T]}$$

where $\underline{r}_t = \underbrace{(r_t, ..., r_t)^T}_{m\times}$ is the vector which has r_t in any component. Such a process is called a **state price process** or **market price of risk** . For any such process we define the corresponding measure $\mathbb{P}_\theta^*$ via $\mathbb{P}_\theta^*(A) = \mathbb{E}_\mathbb{P}(1_A \cdot Z_T)$ where $Z_T = \mathcal{E}(\int \theta \cdot d\mathbb{W})$ (here we need that θ_t is pathwise bounded, which follows from the uniform coercitivity of σ_t) and the discussion above shows that $\mathbb{P}_\theta^* \in \mathcal{P}(\mathcal{M}_{genBS})$. Hence we have proven the following theorem :

Theorem 4.1.1. $\mathcal{P}(\mathcal{M}_{genBS}) = \{\mathbb{P}_\theta^* \mid \theta$ *is a state price process* $\}$.

If σ_t does not have full rank, then in general there is no such θ at all. In this case no equivalent martingale measure exists. It can be shown that in this case the market is not arbitrage free. If the σ_t has full rank, then more then one state processes may exist. In this case the market is arbitrage free, but their is no unique fair price for a contingent claim on this market (of course however if traders fix one of these prices everything works fine and no one can take advantage). The best situation is, if σ_t is invertible a.s. for all $t \in [0,T]$. In this case θ_t must be $\sigma_t^{-1} \cdot (\underline{r}_t - b_t)$ and we have :

Theorem 4.1.2. *If* $det(\sigma_t) \neq 0$ *a.s. for all* $t \in [0,T]$ *then we have*

$$\mathcal{P}(\mathcal{M}_{genBS}) = \{\mathbb{P}_\theta^* \,|\theta_t = \sigma_t^{-1} \cdot (\underline{r}_t - b_t)\}$$

and $|\mathcal{P}(\mathcal{M}_{genBS})| = 1$.

In the last case we have a unique equivalent martingale measures and hence fair prices can only be computed by one formula. For a one dimensional model where the processes r_t, b_t and σ_t are constants and a European call option this is the price computed in Theorem 2.7.1. Let us now consider the question, whether arbitrage freeness implies the existence of martingale measures. In fact we can proof the following theorem :

Theorem 4.1.3. *If $\mathcal{M}_{genBS}$ is arbitrage free, then $\mathcal{P}(\mathcal{M}_{genBS}) \neq \emptyset$. Hence $\mathcal{M}_{genBS}$ satisfies the Fundamental Law of Asset Pricing (see def. 2.3.6).*

Proof. Let $\varphi \in \Phi$ be a trading-strategy. Assume there exists $A \subset [0,T] \times \Omega$ s.t. $(\mu \otimes \mathbb{P})(A) > 0$ and

$$\begin{aligned} \sigma^T \cdot \varphi &= 0 \\ \varphi^T \cdot (b - \underline{r}) &\neq 0 \end{aligned}$$

on A, where $\underline{r}$ denotes the vector with all entries equal to r. Let us define a new process ψ_i as

$$\psi^i := \frac{1}{\tilde{X}_t^i} sgn(\varphi^T \cdot (b - \underline{r})) \cdot \varphi^i \cdot 1_A \text{ for } i = 1, .., n$$

and ψ^0 s.t. $\psi = (\psi^0, \psi^1, ..., \psi^n)$ is self-financing and $V_0(\psi) = 0$ (this is always possible). Here sgn denotes the sign function, which is 1, if the argument is greater or equal than zero and -1 else. We have

$$
\begin{aligned}
d(\tilde{V}_t(\psi)) &= \sum_{i=1}^{n} \psi_t^i d\tilde{X}_t^i \\
&= \sum_{i=1}^{n} \psi_t^i \tilde{X}_t^i ((b_t^i - r_t)dt + \sum_{j=1}^{m} \sigma_{ij,t} dW_t^j) \\
&= 1_A \cdot \sum_{i=1}^{n} sgn(\varphi^T \cdot (b - \underline{r}))_t \cdot \varphi^i \frac{1}{\tilde{X}_t^i} \tilde{X}_t^i ((b_t^i - r_t)dt + \sum_{j=1}^{m} \sigma_{ij,t} dW_t^j) \\
&= 1_A \cdot |\varphi_t^T \cdot (b_t - \underline{r}_t)|_d t + 1_A \cdot sgn(\varphi_t^T \cdot (b_t - \underline{r}_t)) \cdot \sum_{j=1}^{n} \underbrace{(\sum_{i=1}^{n} \varphi_t^i \cdot \sigma_{ij,t})}_{=(\sigma^T \cdot \varphi)_j = 0 \, on \, A} dW_t^j \\
&= 1_A \cdot |\varphi_t^T \cdot (b_t - \underline{r}_t)| dt.
\end{aligned}
$$

Clearly $\tilde{V}_t(\psi) = \int_0^t d\tilde{V}_s(\psi) = \int_0^t 1_A \cdot |\varphi^T \cdot (b - \underline{r})| dt \geq 0$ for all $0 \leq t \leq T$ and since $(\mu \hat{\otimes} \mathbb{P})(A) > 0$ and $|\varphi^T \cdot (b - \underline{r})| dt > 0$ on A we have

$$
\begin{aligned}
\mathbb{E}(\tilde{V}_T(\psi)) &= \mathbb{E}(\int_0^T 1_A \cdot |\varphi_t^T \cdot (b_t - \underline{r}_t)| dt \\
&= \int_{[0,T] \times \Omega} 1_A \cdot |\varphi_t^T \cdot (b_t - \underline{r}_t)| d(\mu \otimes \mathbb{P}) > 0.
\end{aligned}
$$

This implies $\mathbb{P}(V_t(\psi) > 0) = \mathbb{P}(\tilde{V}_t(\psi) > 0) > 0$ and ψ is an arbitrage strategy. By our assumption, the market is arbitrage free, such a trading strategy cannot exist. Hence any A as chosen on the beginning of this proof must satisfy $(\mu \hat{\otimes} \mathbb{P})(A) = 0$ and

$$
\sigma^T \cdot \varphi = 0 \Rightarrow \varphi \perp (b - \underline{r}) \text{ a.s.}
$$

Therefore we have $(b - r) \in (ker(\sigma^\top))^\perp = im(\sigma)$ and there exists a process θ s.t.

$$
\sigma \cdot \theta = b - \underline{r}
$$

(the argumentation here is not really precise, one has to show that this θ can be chosen as a adapted (in fact predictable) process, this is very technical). As is Theorem 4.1.1. we have $\mathbb{P}^*_{-\theta} \in \mathcal{P}(\mathcal{M}_{genBS})$.

$\square$

Let us now consider the question of completeness. We assume that $det(\sigma_t) \neq 0$ a.s. for all $t \in [0,T]$, hence $n = m$ and there exists a unique equivalent martingale measure $\mathbb{P}^* = \mathbb{P}^*_\theta$ and by Proposition 3.5.1

$$\mathbb{W}^*_t = \mathbb{W}_t - \int_0^t \theta_s ds$$

is a Brownian motion under $\mathbb{P}^*$. We also have

$$d\tilde{X}^i_t = \tilde{X}^i_t \sum_{j=1}^{n} \sigma_{ij,t} dW^{j*}_t.$$

Let us consider a contingent claim g. We assume that

$$\mathbb{E}_{\mathbb{P}}(e^{-\int_0^T r_t dt} \cdot g) < \infty. \tag{4.5}$$

The last condition is equivalent to $\mathbb{E}_{\mathbb{P}^*}(e^{-\int_0^T r_t dt} \cdot g) < \infty$ by the explicit form of $\mathbb{P}^*$. We consider the martingale (with respect to the Brownian filtration)

$$\tilde{g}_t := \mathbb{E}_{\mathbb{P}^*}(e^{-\int_0^T r_t dt} \cdot g | \mathcal{F}_t). \tag{4.6}$$

It follows from Theorem 3.7.1 that there exists a predictable $\mathbb{R}^n$ valued stochastic process H such that

$$\tilde{g} = \int H \cdot d\mathbb{W}. \tag{4.7}$$

We define the matrix valued process $\tilde{\sigma}$ as

$$\tilde{\sigma}_{ij,s} := X^i_s \sigma_{ij,s} \ , \ 1 \leq i,j \leq n. \tag{4.8}$$

Clearly $det(\tilde{\sigma}_s) = (\prod_{i=1}^n X^i_s) \cdot det($
$sigma_s) \neq 0$ a.s. for all $t \in [0,T]$. We define $\tilde{\varphi}_s = (\varphi^1_s, ..., \varphi^n_s)$ by

$$\tilde{\varphi}_s := H_s \cdot \tilde{\sigma}_s^{-1} \quad \Leftrightarrow \quad \varphi_s^i = \sum_{k=1}^{n} H_s^k \tilde{\sigma}_{ki,s}^{-1}\,, 1 \leq i \leq n. \tag{4.9}$$

Then we have

$$\begin{aligned}
\sum_{i=1}^{n} \varphi_s^i d\tilde{X}_s^i &= \sum_{i=1}^{n} \varphi_s^i \sum_{j=1}^{n} \underbrace{\tilde{X}_s^i \sigma_{ij,s}}_{\tilde{\sigma}_{ij,s}} dW_s^{j*} \\
&= \sum_{j=1}^{n} \sum_{k=1}^{n} H_s^k \underbrace{(\sum_{i=1}^{n} \tilde{\sigma}_{ki,s}^{-1} \tilde{\sigma}_{ij,s})}_{=\delta_{kj}} dW_s^{*j} \\
&= \sum_{j=1}^{n} H_s^j dW_s^j = d\tilde{g}_s.
\end{aligned}$$

we define $\varphi_s^0 := \tilde{g}_s - \sum_{j=1}^n \varphi_s^j \tilde{X}_s^j$. We show that $\varphi = (\varphi^0, \varphi^1, ..., \varphi^n)$ is a self-financing trading-strategy. We have

$$\tilde{V}_t(\varphi) = \varphi_t \cdot \tilde{X}_t = (\tilde{g}_t - \sum_{j=1}^{n} \varphi_t^j \tilde{X}_t^j) \cdot 1 + \sum_{j=1}^{n} \varphi_t^j \tilde{X}_t^j = \tilde{g}_t$$

which then implies

$$d\tilde{V}_t(\varphi) = d\tilde{g}_t = \sum_{i=1}^{n} \varphi_t^i d\tilde{X}_t^i \underbrace{=}_{d\tilde{X}_t^0 = 0} \varphi_t d\tilde{X}_t.$$

Hence φ is self-financing. Last but not least we have

$$V_T(\varphi) = \tilde{V}_T(\varphi) \cdot X_T^0 = \tilde{g}_T \cdot e^{\int_0^T r_t dt} = \mathbb{E}_{\mathbb{P}^*}(g \cdot e^{-\int_0^T r_t dt} | \mathcal{F}_T) \cdot e^{\int_0^T r_t dt} = g.$$

Since φ also satisfies conditions (1) and (3) on page 74 we have a found a hedge $\varphi \in \Phi$ for the contingent-claim g. hence we have proven the following theorem :

Theorem 4.1.4. *If in the generalized Black-Scholes financial market model* $det(\sigma_t) \neq 0$ *a.s. for all* $t \in [0, T]$*, then this model is complete in the sense that any contingent claim* g *which satisfies* $\mathbb{E}_{\mathbb{P}^*}(e^{-\int_0^T r_t dt} \cdot g) < \infty$ *can be hedged by a trading strategy* $\varphi \in \Phi$.

Corollary 4.1.1. *If* $|\mathcal{P}(\mathcal{M}_{genBS})| = 1$ *then* $\mathcal{M}_{genBS}$ *is complete in the sense above.*

The inverse implication is left as an exercise.

Exercise 4.1.1. *Show if* $\mathcal{M}_{genBS}$ *is complete, then* $|\mathcal{P}(\mathcal{M}_{genBS})| = 1$.

The following theorem summarizes the results :

Theorem 4.1.5. *The following two statements are equivalent :*

1. $|\mathcal{P}(\mathcal{M}_{genBS})| = 1$
2. $\mathcal{M}_{genBS}$ *is complete.*

4.2 A simple stochastic Volatility Model

Let us assume that on $(\Omega, \mathcal{F}, \mathcal{P})$ there exists a 1-dimensional Brownian motion (W_t) and an independent process (σ_t) given by

$$\sigma_t := \begin{cases} 2 & 0 \leq t \leq \tau \quad \text{with probability } 1 \\ 1 & \tau < t \leq T \quad \text{with probability } 1/2 \\ 3 & \tau < t \leq T \quad \text{with probability } 1/2 \end{cases}$$

where $\tau : \Omega \to [0, T]$ is a stopping time and $\mathcal{F}_t := \overline{\sigma(W_s, \sigma_s | 0 \leq s \leq t)_+}$. Because of the independence of W and σ (W_t) is also a Brownian motion with respect to $\mathcal{F}_t$ (check Definition 1.3.1). Define

$$\begin{aligned} X_t^0 &:= 1 \\ X_t^1 &:= X_0^1 \cdot exp(\int_0^t \sigma_s dW_s - \frac{1}{2} \int_0^t \sigma_s^2 ds) = \mathcal{E}(\int \sigma dW)_t. \end{aligned}$$

we consider these as tradeable assets (Bond and Stock) and the volatility σ_t as nontradeable asset. The trading-strategies are defined via the same conditions as in section 4.1. We can think of this market as a standard Black-Scholes market where at random time τ the market conditions change (either the market gets more volatile or it calms down). However, since the numeraire is chosen to be 1 we have $\tilde{X} = X$ and since by Remark 3.4.1 $\mathbb{P} \in \mathcal{P}(\mathcal{M})$. On the other side, assume we have a measure $\mathbb{P}^*$ such that

$$
\begin{aligned}
\mathbb{P}^*_{\mathcal{F}^W_T} &= \mathbb{P}_{\mathcal{F}^W_T} \\
\mathbb{P}^*(\sigma_T = 1) &= p \\
\mathbb{P}^*(\sigma_T = 3) &= 1 - p
\end{aligned}
$$

for any $0 < p < 1$ then $\mathbb{P}^* \sim \mathbb{P}$ and $\mathbb{P}^* \in \mathcal{P}(\mathcal{M})$ ($\mathcal{E}(\int \sigma dW)$ remains a martingale under $\mathbb{P}^*$. It is not hard to show, that the value p determines the martingale measure uniquely and hence that

$$\mathcal{P}(\mathcal{M}) \cong (0,1)$$

where the right hand side denotes the open interval. Since martingale measures exist, the market above is arbitrage free. Is it complete ? For this, let us consider the contingent claim $g = \sigma_T$. Since (σ_t) is also a martingale with respect to $\mathbb{P}$ (easy exercise !) we have

$$\sigma_t = \mathbb{E}(g|\mathcal{F}_t).$$

Assume now that g could be hedged, i.e. there would exist $\varphi \in \Phi$ such that $\varphi_T X_T = \sigma_T$. Since φ must be self-financing we have $\varphi_t X_t = \int_0^t \varphi_s dX_s$ and this is a martingale under $\mathbb{P}$. Hence

$$\varphi_t X_t = \mathbb{E}(\varphi_T X_T|\mathcal{F}_t) = \mathbb{E}(g|\mathcal{F}_t) = \sigma_t.$$

Since $\Delta(\int \sigma dW_s)_s = \sigma_s(\Delta W)_s \equiv 0$ we have that (X_t) is continuous.

Hence $\Delta(\varphi_t X_t) = \Delta(\int \varphi_s dX_s)_s = \varphi_s(\Delta X)_s \equiv 0$ and $\varphi_t X_t = \sigma_t$ is continuous with respect to t. This is a contradiction, since σ_t jumps at time τ. Hence such a trading strategy φ can not exist and $\mathcal{M}$ is not complete. From a rational point of view, the non-completeness is caused by the extra randomness induced by the volatility and the fact, that this randomness cannot be traded. The same situation occurs in the more elaborate stochastic volatility models in the next section.

Exercise 4.2.1. *Study how the different values of p determine different values for the fair price of a European call option.*

4.3 Stochastic Volatility Model

let us assume that on the complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ there is given a 2-dimensional Brownian motion $\mathbb{W}_t = (W_t^1, W_t^2)$ and that $\mathcal{F}_t = \mathcal{F}_t^{\mathbb{W}}$ is the Brownian filtration. Consider a market model consisting of

$$\begin{aligned} \text{Bond (Bank account)} &: B_t \\ \text{Stock} &: S_t \\ \text{Volatility} &: \sigma_t \end{aligned}$$

where the Bond and the Stock is tradeable, but the volatility is not. For the trading strategies we assume the same conditions as in section 4.1. To make this model more precise, we assume

$$\begin{aligned} B_t &= exp(rt) \\ S_t &= S_0 \cdot exp(\int_0^t (b - \frac{1}{2}\sigma_t^2)dt + a' \int_0^t \sigma_t dW_t^1 + a \cdot a \int_0^t \sigma_t dW_t^2) \end{aligned}$$

and σ_t is predictable and satisfies

$$df(\sigma_t) = \beta(\sigma_t)dt + \alpha(\sigma_t)dW_t^2\ ,\ \sigma_0 = const. \tag{4.10}$$

where $f : \mathbb{R} \to \mathbb{R}$ is a strictly increasing, two times differentiable function and α, β are at least continuous functions (we will later also make assumptions on α and β). If (4.10) has a solution (a process σ_t which satisfies (4.10)) then we get σ_t simply by $\sigma_t = f^{-1}(f(\sigma_t))$. Notice that we can also write equation (4.10) in terms of $d\sigma$ by application of the Ito-formula. The thing is, that sometimes using such an f the equation looks much nicer. In most cases f will be either $f(x) = x$ or $f(x) = x^2$. The existence of such a σ_t in general depends on the functions β, α and f. The following theorem gives us sufficient conditions for existence.

Theorem 4.3.1. *Under the assumptions above, there exists a predictable process σ_t which satisfies (4.10) if*

1. *$f(x) = x$ and α, β are Lipschitz continuous and satisfy the growth condition*
$$\alpha(x)^2 + \beta(x)^2 \leq K \cdot (1 + x^2)$$
for a constant $K > 0$.

2. *$f(x) = x^2$, $\beta(x) = \kappa(\nu - x^2)$ and $\alpha(x) = \theta \cdot x$ for constants κ,ν and θ.*

In both cases σ_t is unique up to indistinguishability.

The following is a list of the most famous stochastic volatility models :

$$\begin{aligned}
1.)\ d\sigma_t &= \kappa(\nu - \sigma_t)dt + \theta dW_t^2 \text{ Ornstein-Uhlenbeck model} \\
2.)\ d\sigma_t &= \kappa\sigma_t dt + \theta\sigma_t dW_t^2 \text{ geometric BM model} \\
3.)\ d\sigma_t &= \sigma_t^{-1}(\nu - \kappa\sigma_t^2)dt + \theta dW_t^2 \text{ simple Heston model} \\
4.)\ d\sigma_t^2 &= \kappa(\nu - \sigma_t^2)dt + \theta\sigma_t dW_t^2 \text{ Heston model .}
\end{aligned}$$

Clearly the coefficients of 1.) and 2.) satisfy the assumptions in Theorem 4.3.1 (1) and (4) corresponds exactly to the second case of Theorem 4.3.1. The theorem cannot be applied to equation 3.) however, since the function $x \to \frac{1}{x}$ is not Lipschitz continuous. Nevertheless equation 3.) can be solved by using the Girsanov theorem. Using the same method one can compute an explicit solution for 1.). We leave this as an Exercise. Let us now consider the question of arbitrage freeness of this market and as always in this case start the quest for equivalent martingale measures. As in section 4.1 one has to find a 2-dimensional process (θ_s) s.t. for $\tilde{Z} = \int \theta_s d\mathbb{W}_s = \int \theta_s^1 dW_s^1 + \int \theta_s^2 dW_s^2$ we have

$$\tilde{S} = M - [M, \tilde{Z}]$$

where M denotes the martingale part of $\tilde{S}_t$ under $\mathbb{P}$. We have

$$d\tilde{S}_t = S_t(b-r)dt + S_t a' \sigma_t dW_t^1 + S_t a \sigma_t dW_t^2$$

and hence $dM_t = S_t a' \sigma_t dW_t^1 + S_t a \sigma_t dW_t^2$. This implies

$$\begin{aligned} d\tilde{S}_t &= dM_t - d[M, \int \theta_s dW_s]_t \\ &= S_t a' \sigma_t dW_t^1 + S_t a \sigma_t dW_t^2 - (S_t a' \sigma_t \theta_t^1 + S_t a \sigma_t^2 \theta_t^2)dt. \end{aligned}$$

Uniqueness of the FV part shows

$$S_t \cdot (b - r) = -(S_t a' \sigma_t \theta_t^1 + S_t a \sigma_t^2 \theta_t^2)$$

which implies that

$$\frac{r-b}{\sigma_t} = a' \theta_s^1 + a \theta_s^2.$$

Hence we get (changing from θ to $-\theta$)

Theorem 4.3.2. *In the stochastic volatility models from above, we have*

$$\mathcal{P}(\mathcal{M}) = \{\mathbb{P}_\theta^* | \frac{d\mathbb{P}_\theta^*}{d\mathbb{P}} = \mathcal{E}(-\int \theta d\mathbb{W}) \textit{ s.t. } \frac{r-b}{\sigma_t} = a'\theta_s^1 + a\theta_s^2\}.$$

Under the assumption that we have chosen an equivalent martingale measure for our stochastic volatility model, let us now discuss, how to compute fair prices, when there is no closed form solution for the volatility process. The keywords are approximation and simulation. To solve the stochastic differential equation we first approximate its solution in the same way as in the case of ordinary differential equation (Euler method) and then simulate the approximated solution and use the Monte Carlo method to compute an approximation of the fair price of a contingent claim. The approximation procedure we discuss here is called the stochastic Euler scheme. This is the easiest approximation scheme. For more elaborate schemes see **[?]**. Assume now that X is given as the solution of

$$dX_t = \beta(X_t)dt + \alpha X_t dW_t,\ X_0 = s,\ t \in [0,T].$$

Let us assume that $N >> 1$ so that $\Delta := \frac{T}{N} << 1$. Then

$$\begin{aligned} \int_t^{t+\Delta} \beta(X_s)ds &\approx \beta(X_t)\Delta \\ \int_t^{t+\Delta} \alpha(X_s)dW_s &\approx \alpha(X_t)\underbrace{(W_{t+\Delta} - W_t)}_{\sim\mathcal{N}(0,\Delta)} \end{aligned}$$

where the second factor on the right side of the second equation is a standard normally distributed with expectation zero and variance Δ distributed random variable. Hence we get

$$X_{(n+1)\Delta} \approx X_{n\Delta} + \beta(X_{n\Delta})\Delta + \alpha(X_{n\Delta})(W_{(n+1)\Delta} - W_{n\Delta}).$$

We use this relationship to implement a recursive scheme. Define

$$\begin{aligned} \hat{X}_0^N &:= x \\ \hat{X}_{(n+1)\Delta}^N &:= \hat{X}_{n\Delta}^N + \beta(\hat{X}_{n\Delta}^N)\Delta + \alpha(\hat{X}_{n\Delta}^N)Z_n \\ Z_n \quad i.i.d &\sim \mathcal{N}(0,\Delta) \\ n &= 1,..,N. \end{aligned}$$

We could now define the process $\hat{X}^N$ piecewise constant. This leads to a non continuous process. If we want our approximation to be continuous we can do this with linear interpolation : For $t \in [i\Delta, (i+1)\Delta)$ we define

$$\hat{X}_t^N := \hat{X}_{i\Delta}^N + \frac{t - i\Delta}{\Delta}(\hat{X}_{(i+1)\Delta}^N - \hat{X}_{i\Delta}^N).$$

In any way, this procedure gives us a sequence of stochastic processes $\hat{X}^N$ that can easily be simulated on the computer. We would wish, that if $N \to \infty$ we have $\hat{X}^N \to X$ in some sense (a.s., ucp, in probability for the final value...). The following Theorem tells us more.

Theorem 4.3.3. *Under the assumptions of Theorem 4.3.1 we have*

$$\mathbb{E}(|X_T - \hat{X}_T^N|) \leq K \cdot \sqrt{\frac{1}{N}}$$

for some constant $K > 0$. Therefore $\hat{X}_T^N \xrightarrow{L^1} X_T$. We speak of **strong convergence of order 1/2**. *Furthermore for each function $g : \mathbb{R} \to \mathbb{R}$ of polynomial growth rate and only finitely many discontinuities, we have*

$$\mathbb{E}(g(X_T)) - \mathbb{E}(g(\hat{X}_T^N)) \leq K \cdot \frac{1}{N}$$

for some other constant $K > 0$. In this case we speak of **weak convergence of order 1**.

We do not give a proof of this theorem. Clearly we think of g from above as a contingent claim. Given a stochastic volatility model on begins by simulation of the process (σ_t) which results in a process $(\hat{\sigma}_t)$. This process is then used in a second simulation for the stock price process :

$$\hat{S}_{i+1}^N := \hat{S}_i^N + \hat{S}_i^N \cdot r \cdot \Delta + \hat{S}_i^N \cdot \hat{\sigma}_{i\Delta} Z_i'$$

where Z_i' i.i.d $\sim \mathcal{N}(0, \Delta)$ and independent from the first Z_n used for the simulation of (σ_t) (this corresponds to the case a = 0). Then as above we have :

$$\mathbb{E}(g(S_T)) - \mathbb{E}(g(\hat{S}_T^N)) \leq K \cdot \frac{1}{N}$$

for some constant $K > 0$ and finally we have

Theorem 4.3.4. *Under the assumptions from above :*

$$\lim_{N\to\infty} e^{-rT}\mathbb{E}(g(\hat{S}_T^N)) \to \textit{fair price of } g.$$

The last step of the simulation is the computation of the expectation $\mathbb{E}(g(\hat{S}_T^N))$. This is done by the so called Monte Carlo method, which relies on the strong law of large numbers. One simulates $\hat{S}_T^N$ n-times for $n >> 1$ and gets realizations $\hat{S}_T^{N,i}$, $i = 1, .., n$. By the strong law of large numbers

$$\frac{1}{n} \cdot \sum_{i=1} n g(\hat{S}_T^{N,i})$$

is a good approximation for the fair price of g. This method can also be used to price options in the standard Black Scholes model (which by the way is a stochastic volatility model with $d\sigma_t = 0, \sigma_t \equiv \sigma$ to compute fair prices for options where there is no closed form solution available. A good "practical " exercise is :

Exercise 4.3.1. *Check the quality of the Monte Carlo method by applying it to the standard Black Scholes model and the European call option and compare the result to the explicit solution of section 2.7.*

In general the quality of this method depends very much on the underlying model and on the smoothness of the pay-out function g.

4.4 The Poisson Market Model

Let $(\Omega, \mathcal{F}, \mathbb{P})$ still denote a complete probability space together with a filtration $(\mathcal{F}_t)$ which satisfies the usual conditions.

Definition 4.4.1. *An $\mathbb{Z}$-valued rcll process (N_t) is called a* **Poisson process** *with* **intensity** λ *if :*

1. $N_0 = 0$
2. $N_t - N_s$ *independent of* $\mathcal{F}_s$
3. $N_t - N_s \sim P(\lambda \cdot (t-s))$ *(Poisson distributed with parameter* $\lambda \cdot (t-s)$*).*

One should compare this definition with the definition of Brownian motion.Besides that the Poisson-process is not continuous, only the third item has been changed, using the Poisson distribution instead of the normal distribution. We mentioned in section 1.3. that the Brownian motion can be thought of some kind of random-walk with very small step-sizes and models the movement of a small particle in some fluid. Let us also give a motivation for the Poisson process. For this let T_i be a sequence of exponentially with parameter λ distributed random variables (i.e. $\mathbb{P}(T_i \leq t) = 1 - e^{-\lambda t}$ for $t \geq 0$ and 0 else). We think of T_i as the time passing between two similar random events (for example the arrival of a customer at some shop). The $S_n := \sum_{i=1}^n T_i$ is the time when the n-th event happens. We define

$$N_t := max\{n | S_n \leq t\}$$

as the number of events that happen before time t. One can show that (N_t) as defined above is a Poisson process with intensity λ. From this description one can also see, that a Poisson process possesses only jumps of size 1. The following is an easy but useful exercise :

Exercise 4.4.1. *Let (N_t) be a Poisson process of intensity λ. Show $(N_t - \lambda t)$ is a martingale.*

It is clear from the definition, that any Poisson process (N_t) is increasing and hence of finite variation on compacts and also pure quadratic jump ($[N]^c = 0$). Let us consider the following market model :

$$
\begin{aligned}
X_t &= \begin{pmatrix} B_t \\ S_t \end{pmatrix} = \begin{pmatrix} \text{Bond} \\ \text{Stock} \end{pmatrix}. \\
B_t &\equiv 1 \\
S_t &= S_0 \cdot exp(bN_t - \mu t)
\end{aligned}
$$

where $b, \mu > 0$ and (N_t) is a Poisson process with intensity λ. For the set of trading strategies we suppose the same conditions as in section 4.1. We call this market model $\mathcal{M}_{Poisson}$. As for any market model, we are interested whether it is arbitrage free or not. To answer this question we define a process Y as

$$Y_t := (exp(b)_1)N_t - \mu t).$$

This process is also rcll, has finite variation on compacts and therefore satisfies $[Y]^c = 0$. We have (see Def. 3.4.2)

$$
\begin{aligned}
\mathcal{E}(Y)_t &= e^{Y_t} \cdot \prod_{0<s\leq t} (1 + (\Delta Y)_s) \cdot exp(-(\Delta Y)_s) \\
&= e^{(exp(b)-1)N_t} \cdot e^{-\mu t} \prod_{0<s\leq t} exp(b) \cdot (exp(-(exp(b) - 1))) \\
&= e^{(exp(b)-1)N_t} \cdot e^{-\mu t} exp(b \cdot N_t) \cdot ee^{-(exp(b)-1)N_t} \\
&= exp(bN_t - \mu t).
\end{aligned}
$$

Here we used $N_t = \sum_{0<s\leq t} (\Delta N)_s$ as well as $(\Delta Y)_s = (exp(b) - 1)(\Delta N)_s$ and hence $(\Delta N)_s \neq 0 \Leftrightarrow (\Delta Y)_s \neq 0$. This shows that

$$S_t = S_0 \mathcal{E}(Y).$$

If we can now find a measure $\mathbb{P}^* \sim \mathbb{P}$ under which Y is a local martingale, then also S is a local martingale under $\mathbb{P}^*$. This follows from proposition 3.4.3. To compute such a $\mathbb{P}^*$ we need a version of the Girsanov Theorem for Poisson processes.

Theorem 4.4.1. *Let* (N_t) *be a Poisson process with intensity* λ *under the measure* $\mathbb{P}$ *and* $T > 0$. *Define a new measure* $\mathbb{P}^*$ *via*

$$\frac{dP^*}{dP} = \frac{\lambda}{\lambda^*}^{N_T} e^{(\lambda-\lambda^*)}T.$$

Then $(N_t)_{t\in[0,T]}$ *is a Poisson process on* $[0,T]$ *under* $\mathbb{P}^*$ *with intensity* λ^*.

Proof. [**?**], page 246. □

Let us now define $\lambda^* := \mu/(exp(b) - 1)$ and define $\mathbb{P}^*$ with respect to this λ^* as in Theorem 4.4.1. Then under $\mathbb{P}^*$ (N_t) is a Poisson process with intensity λ^*. Hence it follows from Exercise 4.4.1. that $(N_t - \lambda^* t)_{t\in[0,T]}$ is a $\mathbb{P}^*$ martingale. Since $Y_t = (exp(b) - 1)N_t - \mu t = (exp(b) - 1)(N_t - \lambda^* t)$ it follows that also Y is a martingale under $\mathbb{P}^*$ and as mentioned before that $\tilde{X} = X$ is a martingale under $\mathbb{P}^*$. Hence $\mathbb{P}^* \in \mathcal{P}(\mathcal{M}_{Poisson})$ and $\mathcal{M}_{Poisson}$ is arbitrage free. One can also show that $\mathcal{M}_{Poisson}$ is complete. For this one needs some kind of martingale representation theorem for Poisson processes (we leave this out). Also $\mathcal{P}(\mathcal{M}_{Poisson}) = \{\mathbb{P}^*\}$. Let us now consider a European call option $g = (S_T - K)^+$ in $\mathcal{M}_{Poisson}$ where K denotes the strike price. For the fair price we have

$$\begin{aligned}\pi &= \mathbb{E}^*_{\mathbb{P}}(g) \\ &= \mathbb{E}^*_{\mathbb{P}}((S_0 \cdot exp(bN_T - \mu T) - K)^+) \\ &= \sum_{n=0}^{\infty} \frac{1}{n!}(\lambda^* T)^n (S_0 \exp(bn - \mu T) - K)^+.\end{aligned}$$

The following exercise is very interesting. It should be solved by experimenting on the computer.

Exercise 4.4.2. *In the standard Black / Scholes model take parameters* $S_0 = 10$*,* $\sigma = 0.4$*,*$r = 0.025$ *and* $T = 5$*. Compute the fair price of a*

European call with strike price $K = 10$ *with the formula in section 2.7. What parameters* λ, b *and* μ *should one take to get approximately the same price for the same option in the Poisson market model. Then look what happens if you slightly change* K.

The “pure “ Poisson market model is not of practical use but good to demonstrate the general concept. Nevertheless it is sometimes used in combination with the standard Black-Scholes model.

Chapter 5

Portfolio Optimization

5.1 Introduction

Consider a financial market $\mathcal{M}$. Up to now we were mainly concerned in questions like how to characterize arbitrage freeness, completeness and how to compute fair prices for contingent claims. A different question is how to choose a trading strategy that in some sense performs in an optimal way. Optimality will be defined via a so called utility function U which may depend on the terminal wealth of the portfolio but also on possible consumption. It is clearly that a greater terminal wealth should have a greater utility, but the utility function should also pay respect to the fact that trading in financial markets bears risks. Also different traders may use different utility functions corresponding to their individual attitude towards risk. However there are some quite general rules :

1. more money is always better (has a greater utility) $\Rightarrow U$ is monotonically increasing

2. if your wealth is 1000 Euro your individual increase in utility from gaining another 1000 Euro is much bigger than if your wealth would be 1000000 Euro (change in utility can be understood as what effect has the gain on your life) $\Rightarrow U'(x)$ is monotonically decreasing

3. if your wealth is infinite (means you already have everything) you utility can no longer increase $\Rightarrow \lim_{x\to\infty} U'(x) = 0$ and vice versa, if your wealth is zero (you have nothing) then your life is changed dramatically, by any gain $\Rightarrow \lim_{x\to\infty} U'(x) = 0$

In this section we do only consider the generalized Black-Scholes model from section 4.1 in the case where $det(\sigma_s) \neq 0$. This is the situation where we have a complete financial market and a unique equivalent martingale measure and do not have to worry which equivalent martingale measure we use. In contrast to what we have done before though, we now also allow the trader to consume some oft he money he has in the market. Mathematically precise :

Definition 5.1.1. *A consumption strategy is a non-negative predictable, real valued stochastic process* $(c_t)_{t\in[0,T]}$ *s.t.*

$$\int_0^T c_t dt < \infty \, \mathbb{P} \text{ a.s.}.$$

Remark 5.1.1. (c_t) *models the consumption-rate,* $\int_0^T c_t dt$ *the money the trader takes from the market for consumption over the time interval* $[0, t]$.

We do now consider pairs (φ, c) consisting of a trading strategy φ which satisfy the conditions 1.) and 2.) in section 4.1 (page 74) and a consumption strategy c where condition 3.) in section 4.1. is replaced by the following self-financing condition :

$$dV_t(\varphi) = \varphi_t \cdot dX_t - c_t dt.$$

Since consumption takes place over the whole time interval $[0, T]$ we must allow a utility function also to depend on a time parameter.

Definition 5.1.2. *1.* $U : (0, \infty) \to \mathbb{R}$ *is called a utility function if it is strictly concave,* C^1 *and* $U'(0) := \lim_{x\to 0} U'(x) = \infty$ *as well as* $U'(\infty) := \lim_{x\to\infty} U'(x) = 0$.

2. *If* $U : [0,T] \times \infty \to \mathbb{R}$ *also depends continuously on time and* $\forall t$ $U(t,\cdot)$ *is a utility function in the sense of 1.) then* U *is also called a utility function.*

It is easy to check that the following functions are utility functions.

1. $U(x) = ln(x)$
2. $U(x) = \sqrt(x)$
3. $U(x) = x^\alpha$ for $0 < \alpha 1$
4. $U(t,x) = e^{-\rho t} U_1(x)$ for $\rho > 0$ and U_1 as in Definition 5.1.2.

Definition 5.1.3. *For a self-financing pair* (φ, c) *consisting of a trading strategy* φ *and a consumption strategy s.t* $V_0(\varphi) = x$ *as well as utility functions* $U_1 : [0,T] \times (0,\infty) \to \mathbb{R}$ *and* $U_2 : (0,\infty) \to \mathbb{R}$ *we define the* **expected utility** *as*

$$J(x,\varphi,c) = \mathbb{E}(\int_0^T U_1(t,c_t)dt + U_2(V_T(\varphi))).$$

We allow the expression in Definition 5.1.3. to be infinite (there is nothing to complain about infinite utility) but for technical reasons we assume that the negative utility has finite expectation, i.e.

$$\mathbb{E}(\int_0^T U_1(t,c_t)^- dt + U_2(V_T(\varphi)^-)) < \infty.$$

The aim of Portfolio Optimization is then to compute

$$\max_{(\varphi,c)\in A'(x)} J(x,\varphi,c)$$

$$A'(x) = \{(\varphi,c) \text{ self-financing pair } V_0(\varphi) = x, \mathbb{E}(\int_0^T U_1(t,c_t)^- dt + U_2(V_T(\varphi)^-)) < \infty\}$$

(φ^*, c^*) such that the maximum is attained .

This problem is also called the **continuous portfolio problem**. In the following two sections we will discuss two methods to solve this problem. The first one is called the **martingale method** the second one the **stochastic control approach**. The first method only works in the the case the market is complete (which is the case here), the second one is from the mathematical point of view more sophisticated but has the advantage that it also works in the non-complete case. This method can easily be adapted to the case where the underlying financial market is not a generalized Black-Scholes market.

5.2 The Martingale Method

The main idea behind this method is to decompose the continuous portfolio problem into two subproblems, where the first one is a static optimization problem, and the second one a representation problem. The existence of a solution of the second problem is guaranteed by the completeness oft he generalized Black-Scholes model which itself is implied by the martingale representation theorem. From this the method got its name. Let us first assume that $c \equiv 0$ and $U_1 \equiv 0$.

Problem 1 : $G(x) := \{g \geq 0 \ \mathcal{F}_T \text{ mb } | \mathbb{E}_{\mathbb{P}^*}(e^{-\int_0^T r_t dt} g) = x, \mathbb{E}_{\mathbb{P}}((U_2(g)^-) < \infty\}$
compute $G^* := \max_{g \in G(x)} \mathbb{E}(U_2(g))$.

Problem 2 : compute $(\varphi, 0) \in A'(x) \text{s.t.} V_T(\varphi) = G^*$.

Before we discuss the solution of Problem 1 let us take a look at the well known **Lagrange multiplier method**, which helps us, to compute extremal points of a function under some subsidiary condition. Assume $f : \mathbb{R}^n \to \mathbb{R}$ is strictly concave and $g : \mathbb{R}^n \to \mathbb{R}^k$ is convex, then

$$
\begin{aligned}
x^* &= max\{f(x)|x \in \mathbb{R}^n, g(x) = 0\} \\
&\Leftrightarrow \exists \lambda^* = (\lambda_1^*, ..., \lambda_k^*) \in \mathbb{R}^k \\
\text{s.t. } & \frac{\partial}{\partial x_i} f(x^*) - \sum_{j=1}^{k} \lambda_j^* \frac{\partial}{\partial x_i} g^j(x) = 0 \; i = 1, ..., n \\
& g = (g^1, ..., g_k)^\top \text{ and } g(x^*) = 0 \\
&\Leftrightarrow (x^*, \lambda^*) \text{ is a zero of } \nabla L \text{ where} \\
& L(x, \lambda) = f(x) - \lambda^\top g(x) \text{ is the Lagrange-function}
\end{aligned}
$$

For the application of the Lagrange multiplier method in Problem 1, let us define

$$H_t \;:=\; e^{-\int_0^t r_s ds} \mathbb{E}_{\mathbb{P}}(\frac{d\mathbb{P}^*}{d\mathbb{P}}|\mathcal{F}_t)$$

and the Lagrange-function L as

$$L(g, \lambda) \;:=\; \mathbb{E}(U_2(g) - \lambda \cdot (H_T \cdot g - x)), \; \lambda > 0.$$

Setting the partial derivatives equal to zero we get :

$$
\begin{aligned}
\frac{\partial}{\partial g} L(g, \lambda) &= \mathbb{E}(U_2'(g) - \lambda H_T) = 0 \\
\frac{\partial}{\partial \lambda} L(g, \lambda) &= \mathbb{E}(-(H_T \cdot g - x)) = x - \mathbb{E}(H_T \cdot g) \\
&= x - \mathbb{E}_{\mathbb{P}^*}(e^{-\int_0^T r_t dt} g) = 0.
\end{aligned}
$$

The second equation says exactly, that and hedging strategy φ of$g$ satisfies $V_0(\varphi) = x$. Since $U_2' : (0, \infty) \to (0, \infty)$ is bijective (strictly decreasing and surjective), we can define

$$\tilde{g} := (U_2')^{-1}(\lambda H_T).$$

Obviously $\tilde{g}$ solves the first equation from above. Substituting this in the second equation we get

$$x - \underbrace{\mathbb{E}(H_T(U_2')^{-1}(\lambda H_T))}_{=:\chi(\lambda)} = 0. \tag{5.1}$$

The function $\chi : (0, \infty) \to (0, \infty)$ has the following properties :

Lemma 5.2.1. *χ is continuous and strictly decreasing. Furthermore*

$$\begin{aligned} \chi(0) &:= \lim_{\lambda \to 0} \chi(\lambda) = \infty \\ \chi(0) &:= \lim_{\lambda \to \infty} \chi(\lambda) = 0. \end{aligned}$$

Proof. That χ is continuous follows from the continuity of $(U_2')^{-1}$ and the theorem about dominated convergence. Since H_T is strictly positive and $(U_2')^{-1}$ is strictly decreasing, $\chi(\lambda)$ is also strictly decreasing. Since $(U_2')^{-1}$ shares the behavior for $\lambda \to 0$ respectively $\lambda \to \infty$ with (U_2') so does $\chi(\lambda)$. □

Hence $\chi(\lambda)$ is invertible and we can define

$$Y(u) := \chi^{-1}(u). \tag{5.2}$$

Using this notation we see that (5.1) is equivalent to

$$\chi(\lambda) = x \Leftrightarrow \lambda = \chi^{-1}(x) = Y(x)$$

as well as

$$\tilde{g} = (U_2')^{-1}(Y(x)H_T).$$

Then the pair $(\tilde{g}, Y(x))$ is a zero of ∇L and $\tilde{g}$ is a solution of problem 1. Is it ? Even though the argument above seems to be convincing,

g actually stands for a random variable and $\frac{\partial}{\partial g}$ is not defined. Also it is not clear at all,if the Lagrange multiplier methods works in this context. The computations above are just formal computations which lead us to the right solution. We will later see and proof, that $\tilde{g}$ is indeed the solution of problem 1. From now on we also consider the case $c \neq 0$. We define

$$\begin{aligned} I_2(y) &:= (U_2')^{-1}(y) \\ I_1(y) &:= (U_1')^{-1}(t,y) \\ \chi(y) &:= \mathbb{E}(\int_0^T H_t I_1(t, yH_t)dt + H_T I_2(yH_T)). \end{aligned}$$

A slight modification of the proof of lemma 1 shows that χ above has the same properties as χ in lemma 1. We will later need the following lemma.

Lemma 5.2.2. *Let U be a utility function and $I := (U')^{-1}$. Then*

$$U(I(y)) \geq U(x) + y(I(y) - x),\ \forall 0 < y < x < \infty.$$

Proof. Since U is concave we have

$$U(I(y)) \geq U(x) + U'(I(y))(I(y) - x)) = U(x) + y(I(y) - x).$$

□

Let us now present the main theorem of this section.

Theorem 5.2.1. *Let $x > 0$ and $\chi(y) < \infty$ for all $y > 0$. Then*

$$\begin{aligned} g^* &:= I_2(Y(x) \cdot H_T) \text{ is the optimal wealth} \\ c_t^* &:= I_1(t, Y(x) \cdot H_t) \text{ is the optimal consumption} \end{aligned}$$

and there exists a trading-strategy φ^* *s.t.* $(\varphi^*, c^*) \in A'(x)$ *and* $V_T(\varphi^*) = g^*$.

Proof. For simplicity we assume $U_1 \equiv 0$ and hence $c^* \equiv 0$. By definition we have

$$\mathbb{E}(H_T g^*) = \chi(Y(x)) = x.$$

It follows from the completeness of the generalized Black-Scholes model that there exists $\varphi^* \in \Phi$ s.t. $V_T(\varphi^*) = g^*$ and $V_0(\varphi) = x$. Furthermore it follows from Lemma 5.2.2 that

$$U_2(g^*) \geq U_2(1) + Y(x) \cdot H_T(g^* - 1)$$

and since $a \geq b \Rightarrow a^- \leq b^- \leq |b|$ we have

$$\begin{aligned}
\mathbb{E}(U_2(g^*)^-) &\leq \mathbb{E}(|U_2(1)| + Y(x) \cdot H_T(g^* + 1)) \\
&= |U_2(1)| + Y(x)(\underbrace{\mathbb{E}(H_T g^*)}_{=x} + \mathbb{E}(H_T \cdot 1)) \\
&= |U_2(1)| + Y(x)(x + \underbrace{\mathbb{E}_{\mathbb{P}^*}(e^{-\int_0^T r_s ds})}_{<\infty}) < \infty
\end{aligned}$$

which shows that $(\varphi^*, 0) \in A'(x)$. That the expectation under the risk neutral measure $\mathbb{P}^*$ is finite follows from the uniform boundedness of the interest-rate process r_t, which was an assumption in the generalized Black-Scholes model. Let us now show the optimality of $(\varphi^*, 0)$. We assume that $(\varphi, 0) \in A'(x)$ is arbitrary. Then from Lemma 5.2.2 we deduce that

$$U_2(g^*) \geq U_2(V_T(\varphi)) + Y(x) H_T \cdot (g^* - V_T(\varphi)).$$

Building expectations on both sides we get

$$
\begin{aligned}
\mathbb{E}(U_2(g^*)) &\geq j(x,\varphi,0) + Y(x)\cdot \mathbb{E}(H_T(g^* - V_T(\varphi)) \\
&= J(x,\varphi,0) + Y(x)\cdot (x - \underbrace{\mathbb{E}_{\mathbb{P}^*}(\tilde{V}_T(\varphi)))}_{=x} \\
&= J(x,\varphi,0)
\end{aligned}
$$

where we have used that under the risk neutral measure $\mathbb{P}^*$ the discounted value process $\tilde{V}_T(\varphi))$ is a martingale. Hence $J(x,\varphi^*,0) \geq J(x,\varphi,0)$ and the theorem is proven. $\square$

For computations it is sometimes advantageous to consider the so called **portfolio process**.

Definition 5.2.1. *Let (φ, c) be a self-financing pair consisting of a trading-strategy φ and a consumption process c s.t. $V_t(\varphi) > 0$ a.s. for all $t \in [0,T]$. Define*

$$
\begin{aligned}
\pi_t &:= (\pi_t^1, ..., \pi_t^n)^\top \forall t \in [0,T] \\
\pi_t^i &:= \frac{\varphi_t^i X_t^i}{V_t(\varphi)}.
\end{aligned}
$$

The portfolio-process describes how much relative to the total wealth the trader invests into each asset. Clearly we have

$$
\begin{aligned}
1 - \pi_t^\top \cdot \underline{1} &= 1 - \sum_{i=1}^n \frac{\varphi_t^i X_t^i}{V_t(\varphi)} \\
&= \frac{V_t(\varphi) - \sum_{i=1}^n \varphi_t^i X_t^i}{V_t(\varphi)} \\
&= \frac{\varphi_t^0 X_t^0}{V_t(\varphi)}
\end{aligned}
$$

which shows that π_t also carries information about the $0-$th asset. Assume now that (φ, c) is a self-financing pair. Then

$$
\begin{aligned}
dV_t(\varphi) &= \sum_{i=1}^{n} \varphi_t^i dX_t^i - c_t dt \\
&= \varphi_t^0 \cdot X_t^0 r_t dt + \sum_{i=1}^{n} \varphi_t^i X_t^i (B_t^i dt + \sum_{j=1}^{n} \sigma_{ij,t} dW_t^j) - c_t dt \\
&= (1 - \pi_t^\top \cdot \underline{1}) V_t(\varphi) r_t dt + \sum_{i=1}^{n} V_t(\varphi)\pi : t^i (b_t^i dt + \sum_{j=1}^{n} \sigma_{ij,t} dW_t^j) - c_t dt \\
&= (1 - \pi_t^\top \underline{1}) V_t(\varphi) r_t dt + V_t(\varphi)\pi_t^\top b_t dt + V_t(\varphi)\pi_t^T \sigma_t d\mathbb{W}_t - c_t dt.
\end{aligned}
$$

Reordering terms, we get the so called **Wealth equation** (short notation WE)

$$
\begin{aligned}
dV_t &= [r_t V_t - c_t]dt + V_t \pi_t^\top ((b_t - \underline{r}_t)dt + \sigma_t d\mathbb{W}_t) \\
V_0 &= x
\end{aligned}
$$

It is clear how to compute the portfolio process, given the trading- and consumption strategy. It is not so so clear how one gets the trading strategy back given the portfolio process. The following proposition says that the two concepts “trading strategies “ and “portfolio processes “ are mainly equivalent.

Proposition 5.2.1. *Given any predictable process π s.t. $\int_0^T ||\pi_t||^2 dt < \infty$ $\mathbb{P}$ a.s. and consumption process (c_t) then WE has a solution V which itself determines a trading strategy φ s.t. $V_t(\varphi) = V$ and (φ, c) is self-financing. On the other side, given any self-financing pair, the wealth process $V = V(\varphi)$ is a solution of WE for the associated portfolio process π and $x = V_0(\varphi)$.*

Proof. The second part is clear, the first part also, if we know that (WE) has a solution at all. That this is indeed the case under the assumptions made, follows from some theorem about the existence of solutions of linear stochastic differential equations ([**?**], page 62). □

Definition 5.2.2. *Given a process π and a consumption process c as in proposition 5.2.1, the solution V of (WE) is denoted by $V_t(\pi, c)$ and is called the wealth process corresponding to the portfolio-process π and the consumption process c.*

To the pair (π, c) corresponds according to Proposition 5.2.1 a trading strategy φ. Clearly we have $V_t(\varphi) = V_t(\pi, c)$. Note here that given the trading strategy, it is not necessary for the computation of the wealth process also to know the consumption process. In the forth coming we will switch back and forth between either of the two point of views. The continuous portfolio problem is equivalent to

$$\max_{(\pi,c)\in A'(x)} J(x, \pi, c)$$

by $V_T(\varphi) \to V_T(\pi, c)$ and hence can be formulated completely without using trading strategies. Let us now consider the following example :

As utility functions we take $U_1(t, x) \equiv 0$ (hence $c \equiv 0$) and $U_2(x) = lm(x)$. We have

$$\begin{aligned} U_2'(x) &= \frac{1}{x} \\ \Rightarrow I_2(x) &= (U_2')^{-1}(x) = \frac{1}{x} \\ \chi(y) &= \mathbb{E}(H_T \frac{1}{y \cdot H_T}) = \frac{1}{y} \\ \Rightarrow Y(x) &= \frac{1}{x}. \end{aligned}$$

Hence by Theorem 5.2.1 the optimal terminal wealth is

$$g^* = I_2(Y(x)H_T) = \frac{x}{H_T} \Leftrightarrow V_T(\varphi^*)H_T = x.$$

It follows from the integration by parts formula (Proposition 3.3.3) that

$$dH_t = d(e^{-\int_0^t r_s ds}\mathcal{E}(-\int \theta d\mathbb{W})_t)$$
$$= \quad -H_t\theta_t \cdot d\mathbb{W}_t - H_t r_t dt$$

where $\theta_t = \sigma_t^{-1}(b_t - \underline{R}_t)$ denotes the market price of risk. Using the wealth equation and Proposition 3.3.6 we can compute

$$\begin{aligned} d[V(\varphi^*), H]_t &= -H_t V_t(\varphi^*)\pi_t^\top \sigma_t \theta_t dt \\ &= -H_t V_t(\varphi^*)\pi_t^\top (b_t - \underline{r}_t)dt. \end{aligned}$$

Another application of the Integration by parts formula leads to the following computation :

$$\begin{aligned} d(H_t V_t(\varphi)) &= H_t dV_t(\varphi) + V_t(\varphi)dH_t + d[V_t(\varphi, H]_t \\ &= H_t(r_t V_t(\varphi^* dt + V_t \pi_t^\top ((b_t - \underline{r}_t)dt + \sigma_t d\mathbb{W}_t) \\ &- V_t(\varphi^*)(H_t \theta_t^t op d\mathbb{W}_t H_t r_t dt) \\ &= d[V(\varphi^*), H]_t \\ &= H_t V_t(\varphi^*)(\pi_t^\top \sigma_t - \theta^\top)d\mathbb{W}_t. \end{aligned}$$

Using the assumption in section 4.1 on σ, r and φ one can easily show, that the last expression is not only a local martingale but a martingale under $\mathbb{P}$.Therefore we have that

$$\begin{aligned} H_t V_t(\varphi^*) &= \mathbb{E}(H_T V_T(\varphi^*)|\mathcal{F}_t) \\ &= \mathbb{E}(x|\mathcal{F}_t) \\ &= x. \end{aligned}$$

In particular the process $(H_t V_t(\varphi^*)$ is deterministic which implies that

$$\pi_t^\top \sigma_t - \theta^\top = 0.$$

The last equation is equivalent to

$$\pi_t = (\sigma_t^\top)^{-1}\theta_t$$

which has to be valid for all $t \in [0, t]$. This strategy is the portfolio process of the optimal trading strategy. For the standard Black-Scholes model where b, r and σ are constant one gets

$$\pi_t = \frac{b - r}{\sigma^2}$$

which is constant in time. Note that since the prices of the assets change over time, this doesn't mean that the optimal trading strategy is also constant. In fact it is highly non constant and the trader has to rearrange his portfolio at any time. Unfortunately there is no general method to find the optimal portfolio. Under some assumptions there are methods from Malliavin Calculus (Clark-Ocone formula) which may be used to compute the optimal portfolio process. These methods won't be considered in this course. We will discuss a method, which is more elementary, but needs more assumptions.

Proposition 5.2.2. *Let $x > 0$, $\chi(y) < \infty$ for all $y > 0$ and g^*, c^* be the optimal terminal wealth resp. consumption as in Theorem 5.2.1. Assume there exists $f \in C^{1,2}([0,T] \times \mathbb{R}^d)$ s.t. $f(0) = x$ and*

$$\frac{1}{H_t}\mathbb{E}(\int_t^T H_s c_s^* ds + H_T g^* | \mathcal{F}_t) = f(t, W_t^1, ..., W_t^d)$$

then the optimal portfolio process has the following form

$$\pi_t^* = \frac{1}{V_t(\pi^*, c^*)}(\sigma^\top)^{-1}\nabla_x f(t, W_t^1, ..., W_t^n).$$

Proof. Applying the Ito-formula gives :

$$
\begin{aligned}
\frac{1}{H_t}\mathbb{E}(\int_t^T H_s c_s^* ds + H_T g^* | \mathcal{F}_t) &= f(0) + \int_0^t \frac{\partial f}{\partial s}(s, W_s^1, ..., W_s^d) ds \\
&= + \sum_{i=1}^n \frac{\partial^2 f}{\partial x_i^2}(s, W_s^1, ..., W_s^d) ds \\
&= + \sum_{i=1}^n \frac{\partial f}{\partial x_i}(s, W_s^1, ..., W_s^d) dW_s.
\end{aligned}
$$

On the other side it follows from the self-financing condition that (using the processes H_t instead of switching to the equivalent martingale measure, compare Proposition 2.6.1) that

$$
\begin{aligned}
\frac{1}{H_t}\mathbb{E}(\int_t^T H_s c_s^* ds + H_T g^* | \mathcal{F}_t) &= V_t(\varphi) \\
&\underbrace{=}_{(WE)} x + \int_0^t ((r_s + \pi_s^{*\top}(b_s - \underline{r}_s)) V_s(\pi^*, c^*) - c_s ds) \\
&+ \int_0^t V_s(\pi^*, c^*) \pi_s^{*\top} \sigma_s d\mathbb{W}_s.
\end{aligned}
$$

It then follows from the uniqueness of the martingale part (Theorem 3.6.1) that

$$
\nabla_x f(t, W_t^1, ..., W_t^n)^\top = V_s(\pi^*, c^*) \pi_s^{*\top} \sigma_s
$$

which implies that

$$
\pi_s^* = \frac{1}{V_s(\pi^*, c^*)} (\sigma_s^\top)^{-1} \nabla_x f(t, W_t^1, ..., W_t^n).
$$

□

5.3 The stochastic Control Approach

We consider a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ together with a filtration $(\mathcal{F}_t)_{t\in[t_0,t_1]}$ as well as a stochastic differential equation

$$dX_t = \mu(t, X_t, u_t)dt + \sigma(t, X_t, u_t)d\mathbb{W}_t. \tag{5.3}$$

where $\mathbb{W}_t$ is an m-dimensional Brownian motion and u_t a d-dimensional predictable stochastic process (called the **control**). One should think of, that one can choose the process u on which the solution X of (5.3) depends and hereby "control " X. The main problem of stochastic control is the following :

Given so called **cost functions** L and ψ find u^* such that

$$\begin{aligned}
J(t_0, x, u^*) &= \min_{u\in A(t_0,x)} J(t_0, x, u) \text{ where} \\
J(t_0, x, u) &= \mathbb{E}(\int_{t_0}^{\tau} L(s, X_s^{t_0,x}, u_s)ds + \Psi(\tau, X_\tau^{t_0,x})) \\
A(t_0, x) &= \{u \text{ predictable stochastic process with values in} \\
&\quad U \subset \mathbb{R}^d | \forall x \text{(5.3) has a unique solution } X_t^{t_0,x} \\
&\quad \text{s.t. } X_{t_0}^{t_0,x} = x\}
\end{aligned}$$

where τ denotes a stopping time. We will later make more precise assumptions. Before we discuss how to solve this problem which is also called the **stochastic optimal control problem** we study how it is related to the portfolio optimization problem. Let us consider the Wealth-equation (WE) as a controlled SDE where the control is given by $u = (u^1, u^2) = (\pi, c)$ and

$$dV_t^u = \mu(t, V_t^u, u_t)dt + \sigma(t, V_t^u, u_t)d\mathbb{W}_t \tag{5.4}$$

where

$$\mu(t,x,u) := (r + u_1^\top(b - \underline{r}))x - u_2$$
$$\sigma(t,x,u) := x u_1^\top \sigma$$

and the upper index u in V_t^u indicates the dependence of the solution on the control u. The portfolio optimization problem corresponds to the stochastic optimal control problem via

$$J(t,x,u) := \mathbb{E}(\int_t^T U_1(t, u_t^2)dt + U_2(V_T(\pi, c))$$

where U_1 and U_2 are utility functions and minimum is replaced by maximum. Hence methods for solving the stochastic optimal control problem can be used to solve the portfolio optimization problem. The theory on stochastic optimal control is highly developed, though sometimes a bit technical in its application. To make the mathematics precise we need to make some assumptions and definitions :

$$\begin{aligned}
Q_0 &:= [t_0, t_1) \times \mathbb{R}^n \\
\overline{Q_0} &:= [t_0, t_1] \times \mathbb{R}^n \\
U &\subset \mathbb{R}^d \text{closed} \\
\mu : \overline{Q_0} \times U &\to \mathbb{R}^n \\
\sigma : \overline{Q_0} \times U &\to \mathbb{R}^{n \times m} \\
\mu(\cdot,\cdot,u), \sigma(\cdot,\cdot,u) &\in C^1(\overline{Q_0})\ \forall u \\
||\frac{\partial \mu}{\partial t}|| + ||\frac{\partial \mu}{\partial x}|| &\le C \\
||\frac{\partial \sigma}{\partial t}|| + ||\frac{\partial \sigma}{\partial x}|| &\le C \\
||\mu(t,x,u)|| + ||\sigma(t,x,u)|| &\le C(1 + ||x|| + ||u||)
\end{aligned}$$

Furthermore we need the following definitions :

$$
\begin{aligned}
O &:= \begin{cases} \mathbb{R}^n \text{ or} \\ \text{n-dim manifold with } C^3 \text{ boundary, embedded in } \mathbb{R}^n \end{cases} \\
Q &:= [t_0, t_1) \times O \\
\overline{Q} &:= [t_0, t_1] \times \overline{O} \\
\tau &:= \inf\{t \geq t_0 | (t, X_t) \in Q\} \\
|L(t,x,u)| &\leq C(1 + ||x||^k + ||u||^k) \text{ and} \\
|\psi(t,x)| &\leq C(1 + ||x||^k) \text{ on } \overline{Q} \times U \text{ for a } k \in \mathbb{N}.
\end{aligned}
$$

The **value function** corresponding to the stochastic optimal control problem is defined as follows :

$$
V(t,x) := \inf_{u \in A(t,x)} J(t,x,u) \; \forall (t,x) \in Q.
$$

We denote with a the matrix valued function $a := \sigma\sigma^\top$ and define for each $u \in U$ an operator A^u defined on $C^{1,2}(Q)$ as follows :

$$
(A^u G)(t,x) := G_t(t,x) + \frac{1}{2} \sum_{i,j=1} a_{i,j}(t,x,u) \frac{\partial^2 G}{\partial x_i \partial x_j} + \sum_{i=1}^{n} \mu_i(t,x,u) \frac{\partial G}{\partial x_i}(t,x).
$$

Then the following theorem holds :

Theorem 5.3.1. (Hamilton-Jacobi-Bellmann) *: Let* $G \in C^{1,2}(Q) \cap C(\bar{Q})$ *s.t.* $|G(t,x)| \leq K(1 + ||x||^k)$ *for a constant* $K > 0$ *and* $k \in \mathbb{N}$ *and assume* G *is a solution of*

$$
\begin{aligned}
(\Delta) : \inf_{u \in U} (A^u G(t,x) + L(t,x,u)) &= 0 \; \forall (t,x) \in Q \\
G(t,x) &= \psi(t,x) \forall (t,x) \in \partial^* Q
\end{aligned}
$$

where $\partial^* Q = ([t_0, t_1) \times \partial O) \cup (\{t_1\} \times \bar{O})$*. Then*

1. *$G(t,x) \leq J(t,x,(u_t)) \forall (t,x) \in Q$ and $u \in A(t,x)$*
2. *If $\forall (t,x) \in Q$ there exists $(u_t^*) \in A(t,x)$ s.t.*
   $u_s^* \in argmin_{u \in U}(A^u G(s, X_s^{u^*}) + L(s, X_s^{u^*}, u)) \forall s \in [t, \tau]$
   then

$$G(t,x) = V(t,x) = J(t,x,(u_t^*))$$

Proof. see [?] page 267. □

The following algorithm shows how to apply this theorem to solve the stochastic optimal control problem.

1. solve the minimization problem (Δ) in dependence of the unknown function G and its derivatives
2. let $u_s^* := u^*(s, x, G(s,x), \frac{\partial}{\partial s} G(s,x), \frac{\partial}{\partial x} G(s,x), \frac{\partial^2}{\partial x^2} G(s,x))$
   be the outcome of step 1.), then solve

$$\begin{aligned}(A^{u_t^*} G(t,x) + L(t,x,u_t^*)) &= 0 \, \forall (t,x) \in Q \\ G(t,x) &= \Psi(t,x)\end{aligned}$$

3. check if all assumptions in Theorem 5.3.1 are satisfied

Let us use this algorithm to solve the following portfolio optimization problem in the Black-Scholes market model (for simplicity we assume constant deterministic coefficients) with utility functions

$$\begin{aligned}U_1(t,c) &:= \frac{1}{\gamma} e^{-\beta t} c^\gamma \\ U_2(x) &= \frac{1}{\gamma} x^\gamma\end{aligned}$$

where $\beta > 0$ and $\gamma \in (0,1)$ are constants. The HJB-equation for $V(t,x) := \sup_{u \in A(t,x)} J(t,x,u)$ has the form

$$
\begin{aligned}
0 &= \max_{u^1\in[\alpha_1,\alpha_2]^d, u^2\in[0,\infty)} \{\frac{1}{2}u^{1\top}\sigma\sigma^\top u^1 x^2 \frac{\partial^2}{\partial x^2}V(t,x) \\
&+ ((r+u^{1\top}(b-\underline{r}))x - u^2)\frac{\partial}{\partial x}V(t,x) + U_1(t,u^2) + \frac{\partial}{\partial t}V(t,x)\} \\
V(T,x) &= U_2(x)
\end{aligned}
$$

for given $0 < \alpha_1 < \alpha_2 < \infty$. In the first step we compute the minimum in (Δ) with $L = -U_1$, $\Psi = -U_2$ (for the transition of minima to maxima). For this we build the partial derivatives with respect to u_1 and u_2 and setting them equal to zero gives :

$$
u_t^1 = -(\sigma\sigma^\top)^{-1}(b-\underline{r})\frac{\frac{\partial}{\partial x}V(t,x)}{x\cdot\frac{\partial^2}{\partial x^2}V(t,x)} \tag{5.5}
$$

$$
u_t^2 = (e^{\beta t}\cdot\frac{\partial}{\partial x}V(t,x)^{\frac{1}{\gamma-1}}. \tag{5.6}
$$

In the second step we have to solve the partial differential equation for $V(t,x)$:

$$
\begin{aligned}
0 &= -\frac{1}{2}(b-\underline{r})^\top(\sigma\sigma^\top)^{-1}(b-\underline{r})\frac{(\frac{\partial}{\partial x}V(t,x))^2}{\frac{\partial^2}{\partial x^2}V(t,x)} + r\cdot x\cdot\frac{\partial}{\partial x}V(t,x) \\
&+ \frac{\partial}{\partial x}V(t,x)^{\frac{\gamma}{\gamma-1}}\cdot e^{\frac{\beta t}{\gamma-1}}\cdot\frac{1-\gamma}{\gamma} + \frac{\partial}{\partial t}V(t,x).
\end{aligned}
$$

To solve the last equation we make the following separation approach :

$$
\begin{aligned}
V(t,x) &= f(t)\cdot\frac{1}{\gamma}x^\gamma \\
f(T) &= 1 \text{ (terminal condition)} .
\end{aligned}
$$

Substituting this in the previous equation we get

$$
\begin{aligned}
0 &= [-\frac{1}{2}(b-\underline{r})^\top(\sigma\sigma^\top)^{-1}(b-\underline{r})\cdot 1\gamma - 1 + r]\cdot f(t) \\
&+ \frac{1-\gamma}{\gamma} e^{\frac{\beta t}{\gamma-1}} (f(t))^{\frac{\gamma}{\gamma-1}} + f'(t).
\end{aligned}
$$

Let us define

$$
\begin{aligned}
a_1 &:= -\frac{1}{2}(b-\underline{r})^\top(\sigma\sigma^\top)^{-1}(b-\underline{r})\frac{1}{\gamma-1} + r \\
a_2(t) &:= \frac{1-\gamma}{\gamma} e^{\frac{\beta t}{\gamma-1}}.
\end{aligned}
$$

Using this, we get the following ordinary differential equation for f :

$$
\begin{aligned}
f'(t) &= -a_1\cdot f(t) - a_2 f(t)^{\frac{\gamma}{\gamma-1}} \\
f(T) &= 1.
\end{aligned}
$$

The ODE above is still nonlinear. With the substitution $g(t) = (f(t))^{\frac{1}{\gamma-1}}$ however we get the following linear ODE

$$
\begin{aligned}
g'(t) &= -\frac{a_1}{1-\gamma} g(t) - \frac{a_2(t)}{1-\gamma} \\
g(T) &= 1.
\end{aligned}
$$

This ODE can be easily solve. The solution is

$$
g(t) = e^{\frac{a_1}{1-\gamma}(T-t)} + \frac{1-\gamma}{\gamma(a_1-\beta)}(e^{\frac{a_1-\beta}{1-\gamma}T} - e^{\frac{a_1-\beta}{1-\gamma}t}) \cdot e^{\frac{a_1}{1-\gamma}(T-t)}.
$$

We get the function f as $f(t) = (g(t))^{1-\gamma}$. Substituting this in (5.5) respectively in (5.6) we get for the optimal portfolio respectively the optimal consumption process

$$
\begin{aligned}
\pi_t^* &= \frac{1}{1-\gamma}(\sigma\sigma^\top)^{-1}(b - \underline{r}) \\
c_t &= (e^{\beta t} \cdot f(t))^{\frac{1}{\gamma-1}} \cdot V_t(\pi^*, c^*).
\end{aligned}
$$

The term $V_t(\pi^*, c^*)$ is the value process corresponding to the pair (π^*, c^*). The definition is not recursive as it looks like, since substituting (π^*, c^*) as above in (WE) determines a wealth process uniquely. The third step is now to check, that all the assumption for a application of the HJB theorem are satisfied. This is a very good exercise.

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